Spin representations of Artin's braid group

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Dedicated to the memory of Professor Xiao-Song Lin

Abstract

New linear representations of Artin's group, spin representations and multi-parameter Burau representations, are constructed. They are generalizations of the classical Burau representation.

1 Introduction

When the author was a graduate student with professor Xiao-Song Lin, one problem that I worked on was to search some knot invariants in order to distinguish two different orientations of a knot. In this paper, we report one attempt where the starting point is representations of Artin's braid group. Every geometric braid gives a knot or link by closing, and each knot can be transformed into a closed braid. It was known classically that the Alexander polynomial of a given knot or link can be calculated directly from the image under the reduced Burau representation of a braid, where this braid

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represents a given knot or link. The detailed information in this direction can be found in [2]. The Burau representation admits many different definitions, each in its own way giving some insight [4]. In [5], Long gave a new derivative of the Burau representation by using an action of $Aut(F_n)$ on the representation variety $R = R(F_n, G)$, where G is a compact semisimple Lie group and F_n is the free group of rank n. Once the braid group B_n is viewed as a subgroup of $Aut(F_n)$, for any global fixed curve of B_n inside R, linearization of the action gives a linear representation on the Lie algebra of G^n. We mark each component of a fixed curve, or equivalently, mark each component of the maximal torus of G^n, with different parameters or spin symbols, the action of each braid in B_n will permute those color parameters or spin symbols. We then sum all derivatives (tangent maps) of a braid, which is as a diffeomorphism of the representation variety R, at marked tangent spaces together. This sum will give multi-parameter Burau representations and spin representations on the complex part of the summation of all Lie algebras. For $G = SU(2, C)$, spin representations come up naturally with the action of the Weyl group of G^n on its maximal torus. Our result reported here is also a partial response to "'Open problem 7"' in a recent paper of Birman and Brendle [6]. We also remark that these new representation can not arise from any Hecke algebra, and hope that they may encode more information about the braid group.

The organization of the paper is as follows. In section 2, we recall some related preliminaries and Long's construction. In section 3, we give a new type of representation of the braid groups, and call them spin representations. In section 4, we give multi-parameter representations of the braid groups, and call them multi-parameter Burau representations. We also give a remark about eigenvalues of these representations in order to compare with Jones's Hecke algebra representations. In all cases, we compute out detailed matrix forms for the representations.

2 Preliminaries

For the free group F_n, we fix once and for all a generator set $\{x_1,\ x_2,\ \cdots,\ x_n\}$. The *n-string braid group* B_n, is defined to be the subgroup of $Aut(F_n)$ generated by the automorphisms $\{\sigma_i\ |i = 1,\ 2, \cdots,\ n-1\}$, where the action of σ_i is given by

$$\begin{aligned} x_i &\longmapsto x_{i+1}, \\ x_{i+1} &\longmapsto x_{i+1}^{-1} \cdot x_i \cdot x_{i+1}, \\ x_j &\longmapsto x_j, \quad j \neq i, i+1. \end{aligned}$$

The basic fact about the braid group we need is stated in Artin's theorem [1], that, for any $\beta \in Aut(F_n)$, $\beta \in B_n$ if and only if

$$\begin{aligned} &\beta(x_i) = A_i^{-1} x_{\tau_\beta(i)} A_i,\ 1 \leq i \leq n, \\ &\beta(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n, \end{aligned}$$

where τ_β is the permutation induced by β and A_i is a word in generators $x_1,\ x_2,\ \cdots\ x_n$.

We take the compact semisimple Lie group G to be $SU(2, C)$ for explicit computation purpose. We know that $SU(2, C)$ is the group of unitary matrices of determinant 1, $SU(2, C) = \{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1,\ a, b \in C\}$, which is homeomorphic to the 3-sphere S^3. We compute the tangent bundle as follows. Let $q \in SU(2, C)$, $q = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, is any point, and

$w \in T_q(SU(2,C)$ is a tangent vector, $w = \begin{pmatrix} u & v \\ -\overline{u} & \overline{v} \end{pmatrix}$. There is a differential curve $w(t) : (-1,\ 1) \longrightarrow SU(2,C)$, such that $w(0) = q$ and $\frac{dw}{dt}|_{t=0} = w$. Write $w(t) = \begin{pmatrix} w_1(t) & w_2(t) \\ -\overline{w}_2(t) & \overline{w}_1(t) \end{pmatrix}$, then we have $w_1(t)\overline{w}_1(t) + w_2(t)\overline{w}_2(t) = 1$. Differentiate this equation on both sides, and evaluate at $t = 0$. This gives $\overline{a}u + a\overline{u} + \overline{b}v + b\overline{v} = 0$. So, the tangent bundle of $SU(2,C)$ is

$$T(SU(2,C)) = \{(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} u & v \\ -\overline{v} & \overline{u} \end{bmatrix}) : \overline{a}u + a\overline{u} + \overline{b}v + b\overline{v} = 0, a\overline{a} + b\overline{b} = 1\}.$$

If we take q to be the identity, we get the tangent space, which is the Lie algebra $\mathcal{G}$ of $SU(2,C)$. For any point $q \in SU(2,C)$, the tangent space at this point actually is the image of right translation of $\mathcal{G}$ by q. Furthermore, the Lie algebra $\mathcal{G}$ can be decomposed into two subspaces:

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$$

where

$$\mathcal{G}_1 = \{ \begin{pmatrix} is & 0 \\ 0 & -is \end{pmatrix} : s \in R\}, \quad \mathcal{G}_2 = \{ \begin{pmatrix} 0 & \xi \\ -\overline{\xi} & 0 \end{pmatrix} : \xi \in C\}.$$

We call $\mathcal{G}_1$ the real part, $\mathcal{G}_1$ the complex part. We also need some special inner isomorphisms of $SU(2,C)$ and their linearizations. For any $q \in SU(2,C)$, we have a diffeomorphism $\widetilde{Ad}(q) :\ SU(2,C) \longrightarrow SU(2,C)$ given by $A \longmapsto qAq^{-1}$. Since $\widetilde{Ad}(q)(Id) = Id$, linearizing it at the identity Id, we get an induced isomorphism of the Lie algebra, $Ad(g)(X) = qXq^{-1}$ for $X \in \mathcal{G}$.

Now we recall Long's construction. Set $R = R(F_n, SU(2,C))$, the representation variety of F_n, topologized by the compact open topology. Because

F_n is free, any n-tuple of matrices determines a representation, and any representation determines an n-tuple of matrices. We therefore identify the representation space R with $SU(2,C)^n = SU(2,C) \times SU(2,C) \times \cdots \times SU(2,C)$, and so R is differentiable, which actually is a Lie group. For any $\beta \in Aut(F_n)$, it acts on the representation space R by $\rho \longmapsto \beta^* \rho$, where the image representation is defined by

$$\beta^* \rho(w) = \rho(\beta^{-1}(w)), \quad w \in F_n.$$

Or, view this action in matrix form that $\beta^* : SU(2,C)^n \longrightarrow SU(2,C)^n$ is given by

$$\beta^* : (M_1, M_2, \cdots, M_n) \longmapsto (\beta_*^{-1} M_1, \beta_*^{-1} M_2, \cdots, \beta_*^{-1} M_n).$$

The map β^* is said to be the map associated to β. It is clear that β^* is a diffeomorphis of R although it is not a Lie group homomorphism. The map $Aut(F_n) \longrightarrow Diff(R)$ given by $\beta \longmapsto \beta^*$ is a group homomorphism, and moreover, this is a faithful representation. In order to get a linear representation of a certain subgroup of $Aut(F_n)$, say, the braid group B_n, we have to use smooth structure of the representation variety to linearize the image of B_n under this $*$ map. The parallelization of Lie group $SU(2,C)$, that is, trivialization of tangent bundle of Lie group $SU(2,C)$, makes the linearization possible in the Lie algebra $\mathcal{G}$ alone.

Suppose there is a path of representation $\alpha(t) : (-1,\ 1) \longrightarrow R$, which is fixed by $\beta \in B_n$, namely, $\beta^* \alpha(t) = \alpha(t)$. Then, Long has a theorem that there is a one-parameter family of representation of B_n defined by the composition

$$\rho_t : B_n \longrightarrow Diff(R) \longrightarrow GL(T_{\alpha(t)}(R)),$$

where ρ_t is given by $\rho_t(\beta) = d\beta^*_{\alpha(t)}$.

3 Spin representations

By a spin, we mean an assignment of orientations to each component of the maximal torus $S^1 \times S^1 \times \cdots S^1$ of $SU(2,C)^n$. For example, spin $S = (s_1,\ s_2,\ \cdots,\ s_n)$, where $s_i = \pm 1,\ i = 1,\ 2,\ n$. For each spin S, we define the positive index and the negative index, which are two numbers, $a(S) = \#\{s_i\ |s_i = +1\}$ and $b(S) = \#\{s_i\ |s_i = -1\}$.

Denote $\Lambda_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, where $0 \le \theta \le 2\pi$, then $\Lambda \in SU(2,C)$. Let $\alpha(\theta)$ be a curve given by $\alpha(\theta) = (\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta)$ in R. For any fixed parameter value θ and a spin $S = (s_1,\ s_2,\ \cdots,\ s_n)$, denote $p_\theta = (\Lambda_\theta, \Lambda_\theta, \cdots, \Lambda_\theta)$ and $p_\theta^S = (\Lambda_\theta^{s_1},\ \Lambda_\theta^{s_2},\ \cdots,\ \Lambda_\theta^{s_n})$. If $\beta \in B_n$, β^* is a diffeomorphism of R. By Artin's theorem mentioned above, it is easy to see that

$$\beta^*(A_1,\ A_2,\ \cdots,\ A_n) = (w_1 A_{\tau_\beta(1)} w_1^{-1},\ w_2 A_{\tau_\beta(2)} w_2^{-1},\ \cdots,\ w_n A_{\tau_\beta(n)} w_n^{-1}),$$

where τ_β is the permutation determined by the braid β, w_j is a word on A_1, A_2, $\cdots$, and A_n, here, $j = 1, 2, \cdots, n$. So, we have

$$\beta^*(p_\theta^S) = (\Lambda_\theta^{\tau_\beta(1)},\ \Lambda_\theta^{\tau_\beta(2)},\ \cdots,\ \Lambda_\theta^{\tau_\beta(n)}) =: p_\theta^{\tau_\beta(S)}.$$

Thus, $d\beta^*_{p_\theta^S}$ is a linear transformation from the tangent space $T_{p_\theta^S}(R)$ at p_θ^S to the tangent space $T_{p_\theta^{\tau_\beta(S)}}(R)$ at $p^{\tau_\beta(S)}$. It is clear that $T_{p_\theta^S}(R)$ is the direct sum of Lie algebras of $SU(2,C)$, that is,

$$T_{p_\theta^S}(R) = T_{\Lambda_\theta^{s_1}}(SU(2,C)) \oplus T_{\Lambda_\theta^{s_2}}(SU(2,C)) \oplus \cdots \oplus T_{\Lambda_\theta^{s_n}}(SU(2,C)).$$

Fix two indexes a and b, and $a + b = n$. Consider all spins that have the index $a(S) = a$, and define a vector space marked by spins as follow,

$$V_{a,b} = \bigoplus_{a(S)=a} T_{p_\theta^S}(R).$$

For each fixed θ, there are $\binom{n}{a}$ points on the maximal torus $S^1 \times S^1 \times \cdots S^1$. So, $V_{a,b}$ is the direct sum of tangent spaces at these points.

We now define $\overline{\beta}$ for each $\beta \in B_n$, which is a linear transformation of $V_{a,b}$, by summing all derivatives together, namely,

$$\overline{\beta} = \bigoplus_{a(S)=a} d\beta^*_{p^S_\theta}.$$

If $X \in T_{p^{S_0}_\theta}(R)$ for some spin S_0 with $a(S) = a$ and $b(S) = b$, then $\overline{\beta}(X) \in T_{p^{\tau_\beta(S_0)}_\theta} R$, where τ_β is the permutation determined by β. It is easy to check that, for two braids β_1 and β_2,

$$d\beta^*_{1\, p^{S_1}_\theta} \cdot \beta^*_{2\, p^{S_1}_\theta} = \begin{cases} d(\beta^*_1 \cdot \beta^*_2), & \text{if } S_1 = \tau_{\beta_2}(S_2), \\ 0, & \text{otherwise.} \end{cases}$$

Actually, we get a homomorphism from the braid group to the general linear group of $V_{a,b}$.

Theorem 3.1. *Let $\Phi : \quad B_n \longrightarrow GL(V_{a,b})$ given by $\Phi(\beta) = \overline{\beta}$, then Φ is a group homomorphism.*

Proof. The proof is a simple verification. For any two braids β_1 and β_2, if $\overline{\beta}_1 = \oplus_{a(S)=a}\, d\beta^*_{1,\, p^S_\theta}$ and $\overline{\beta}_2 = \oplus_{a(S)=a}\, d\beta^*_{2,\, p^S_\theta}$, we have

$$\begin{aligned} \overline{\beta}_1 \cdot \overline{\beta}_2 &= \Big(\bigoplus_{a(S)=a} d\beta^*_{1,\, p^S_\theta}\Big) \cdot \Big(\bigoplus_{a(S)=a} d\beta^*_{2,\, p^S_\theta}\Big) \\ &= \bigoplus_{\substack{a(S)=a=a(S') \\ S'=\tau_{\beta_2}(S)}} d\beta^*_{1,\, p^{S'}_\theta} \cdot d\beta^*_{2,\, p^S_\theta} \\ &= \bigoplus_{a(S)=a} d\beta^*_{1,\, p^{\tau_{\beta_2}(S)}_\theta} \cdot d\beta^*_{2,\, p^S_\theta} \\ &= \overline{\beta_1 \cdot \beta_2}. \end{aligned}$$

$\square$

We will consider several special cases in the following subsections. The first two cases seem trivial, but they are basic building blocks for general spin representations, and one of them gives famous Burau representations. In the last subsection, we show that each spin for the maximal torus is determined by an action of the Weyl group, and the action of the Weyl group determines that there are only four types of basic representation matrix blocks.

3.1 Representations with the inverse spin $S = (-1,-1,\cdots,-1)$

Consider a spin with the negative index $b(S) = n$, all negative signs, call it the inverse spin. The $V_{0,n}$ has only one copy of $T_{p_\theta^S}(R)$, specifically, $V_{0,n} = T_{p_\theta^S}(R) = T_{\Lambda_\theta^{-1}}(SU(2,C)) \oplus T_{\Lambda_\theta^{-1}}(SU(2,C)) \oplus \cdots \oplus T_{\Lambda_\theta^{-1}}(SU(2,C))$. We now want to compute the image of any given braid in B_n. It suffices to compute the image of each generator of B_n. It is enough to compute the image of $\sigma_1 \in B_n$ as a representative. We know in this case, $\overline{\sigma}_1 = d{\sigma_1^*}_{p_\theta^S}$. Since $\sigma_1^*(A_1, A_2, \cdots A_n) = (A_1 A_2 A_1^{-1}, A_1, A_3, \cdots A_n)$ for any $A_j \in SU(2,C)$, σ_1^* acts as the identity on the last $n-2$ factors. So, we only consider the action of σ_1^* on the first two components. Suppose w is an element of the Lie algebra $\mathcal{G}$, a tangent vector at the identity Id of $SU(2,C)$. Let's denote a tangent curve in $SU(2,C)$ by $w(t)$ that $w(0) = Id$ and $\frac{d}{dt}w(t)|_{t=0} = w$. For example, $w(t) = e^{wt}$, where $t \in (-1,\ 1)$. A tangent curve to the point $p_\theta^S = (\Lambda_\theta^{-1}, \Lambda_\theta^{-1}, \cdots, \Lambda_\theta^{-1})$ in R can be given by $v(t) = (w(t) \cdot \Lambda_\theta^{-1}, \Lambda_\theta^{-1}, \cdots, \Lambda_\theta^{-1})$. Then, $\sigma_1^*(v(t)) = (w(t) \cdot \Lambda_\theta^{-1} \cdot \Lambda_\theta^{-1} \cdot (w(t) \cdot \Lambda_\theta^{-1})^{-1}, w(t) \cdot \Lambda_\theta^{-1}, \Lambda_\theta^{-1}, \cdots, \Lambda_\theta^{-1})$. To emphasize the fact that the tangent curve is regarded as based at the point $p_\theta^S = (\Lambda_\theta^{-1},\ \Lambda_\theta^{-1},\ \cdots,\ \Lambda_\theta^{-1})$, rewrite the image as

$$((w(t) \cdot \Lambda_\theta^{-1} \cdot w(t)^{-1} \cdot \Lambda_\theta) \cdot \Lambda_\theta^{-1},\ w(t) \cdot \Lambda_\theta^{-1},\ \Lambda_\theta^{-1},\ \cdots,\ \Lambda_\theta^{-1}).$$

Let's now compute the derivative of the first component of this image, differentiating by using product rule:

$$w(t) \cdot w(t)^{-1} = I,$$

$$\frac{d}{dt} w(t) \cdot w(t)^{-1} + w(t) \cdot \frac{d}{dt} w(t)^{-1} = 0,$$

such,

$$\frac{d}{dt} w(t)^{-1} = -w(t)^{-1} \cdot \frac{d}{dt} w(t) \cdot w(t)^{-1};$$

and, so

$$\begin{aligned}
&\frac{d}{dt}(w(t)\Lambda_\theta^{-1} w(t)^{-1}\Lambda_\theta) \\
&= \frac{d}{dt} w(t) \cdot \Lambda_\theta^{-1} w(t)^{-1}\Lambda_\theta + w(t)\Lambda_\theta^{-1} \cdot \frac{d}{dt} w(t)^{-1} \cdot \Lambda_\theta \\
&= (\frac{d}{dt} w(t))\Lambda_\theta^{-1} w(t)^{-1}\Lambda_\theta - w(t)\Lambda_\theta^{-1} w(t)^{-1}(\frac{d}{dt} w(t)) w(t)^{-1}\Lambda_\theta,
\end{aligned}$$

evaluating at $t = 0$, notice $w(0) = Id$ and $\frac{d}{dt} w(t)|_{t=0} = w$, we get

$$\frac{d}{dt}(w(t)\Lambda_\theta^{-1} w(t)^{-1}\Lambda_\theta)|_{t=0} = w - \Lambda_\theta^{-1} w \Lambda_\theta.$$

It is easy to see the second component of the image, which is $\frac{d}{dt} w(t)|_{t=0} = w$. Therefore,

$$d\sigma^*_{1\,p^S_\theta}(w,\, 0, \cdots,\, 0) = (w - \Lambda_\theta^{-1} w \Lambda_\theta,\; w,\; 0,\; \cdots,\; 0).$$

We now consider the action of the derivative on the second component. Let

$$\begin{aligned}
\widetilde{v}(t): \quad (-1,\, 1) \quad &\longrightarrow R \\
t \quad &\longmapsto (\Lambda_\theta^{-1},\; w(t)\Lambda_\theta^{-1},\; \Lambda_\theta^{-1},\; \cdots,\; \Lambda_\theta^{-1}).
\end{aligned}$$

The action of generator σ_1 is

$$\begin{aligned}
\sigma_1^* \widetilde{v}(t) &= \sigma_1^*(\Lambda_\theta^{-1},\; w(t)\Lambda_\theta^{-1},\; \Lambda_\theta^{-1},\; \cdots,\; \Lambda_\theta^{-1}) \\
&= (\Lambda_\theta^{-1} w(t)\Lambda_\theta \cdot \Lambda_\theta^{-1},\; \Lambda_\theta^{-1},\; \cdots,\; \Lambda_\theta^{-1}).
\end{aligned}$$

We evaluate the derivative at $t = 0$, that $\frac{d}{dt}(\Lambda_\theta^{-1}w(t)\Lambda_\theta)|_{t=0} = \Lambda_\theta^{-1}w\Lambda_\theta$. So, we have

$$d\sigma^*_{1\,p^S_\theta}(0,\ w, \cdots,\ 0) = (\Lambda_\theta^{-1}w\Lambda_\theta,\ 0,\ \cdots,\ 0).$$

The space $V_{0,n}$ also has the real part and the complex part. The real part of $V_{0,n}$ is formed by all real parts of their factors: $T_{\Lambda_\theta^{-1}}(SU(2,C)) \oplus T_{\Lambda_\theta^{-1}}(SU(2,C)) \oplus \cdots \oplus T_{\Lambda_\theta^{-1}}(SU(2,C))$, which is an n-dimensional subspace. The complex part of $V_{0,n}$ is formed by all complex parts of their factors, which also is an n-dimensional subspace.

For any vector $X \in \mathcal{G}_1$, i.e, $X = \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix}$ is in the real part of the Lie algebra $\mathcal{G}$, the action of $Ad(\Lambda_\theta^{-1})$ is identity, since $Ad(\Lambda_\theta^{-1})(X) = X$. Therefore,

$$d\sigma^*_{1\,p^S_\theta}(X,\ 0, \cdots,\ 0) = (0,\ X,\ 0,\ \cdots,\ 0),$$

$$d\sigma^*_{1\,p^S_\theta}(0,\ X,\ 0, \cdots,\ 0) = (X,\ 0,\ 0,\ \cdots,\ 0),$$

and

$$d\sigma^*_{1\,p^S_\theta}(0,\ 0,\ X,\ 0, \cdots,\ 0) = (0,\ 0,\ X,\ 0,\ \cdots,\ 0),$$

which is the identity. So, we get the matrix representation for $\overline{\sigma}_1$ on the real part of $V_{0,n}$ as follows

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}.$$

We can conclude that the representation of any element in the braid group B_n on the real part of $V_{0,n}$ is a permutation matrix.

For any element $\begin{pmatrix} 0 & \zeta \\ -\overline{\zeta} & 0 \end{pmatrix}$ in the complex part $\mathcal{G}_2$ of the Lie algebra $\mathcal{G}$, we see, $Ad(\Lambda_\theta^{-1}) \begin{pmatrix} 0 & \zeta \\ -\overline{\zeta} & 0 \end{pmatrix} = \Lambda_\theta^{-1} \begin{pmatrix} 0 & \zeta \\ -\overline{\zeta} & 0 \end{pmatrix} \Lambda_\theta = \begin{pmatrix} 0 & \zeta e^{-2i\theta} \\ -\overline{\zeta} e^{2i\theta} & 0 \end{pmatrix}$.

That is, $\zeta \longmapsto \zeta e^{-2i\theta}$. Set $e^{i\theta} = \lambda$, for any vector X in the complex part $\mathcal{G}_2$, we have

$$\begin{aligned} \overline{\sigma}_1(X,\, 0, \cdots, 0) &= (X - \Lambda_\theta^{-1} X \Lambda_\theta,\, X,\, 0,\, \cdots, 0) \\ &= ((1-\lambda^{-2})X,\, X,\, 0,\, \cdots, 0), \end{aligned}$$

and

$$\begin{aligned} \overline{\sigma}_1(0,\, X,\, 0,\, \cdots, 0) &= (\Lambda_\theta^{-1} X \Lambda_\theta,\, 0,\, \cdots, 0) \\ &= (\lambda^{-2} X,\, 0,\, \cdots, 0). \end{aligned}$$

Therefore, the matrix representation for $\overline{\sigma}_1$ over the complex part of $V_{0,n}$ is

$$\begin{pmatrix} 1-\lambda^{-2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}.$$

Similarly, we can get the matrix representation for other generator $\overline{\sigma}_j$, $1 \leq j \leq n-1$, that is,

$$I_{j-1} \oplus \begin{pmatrix} 1-\lambda^{-2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-j-1}.$$

We denote these matrices by D_j, and this is one family of representation building blocks.

3.2 Burau representations and the spin $S = (+1,+1,\cdots,+1)$

We consider another extreme case where the spin has all positive signs,its indexes $a(S) = n$ and $b(S) = 0$. This is a major case in [5]. For completion, we also mention some details about this case here. The $V_{n,0}$ has only one copy $T_{p_\theta^S}(R)$, that is, $V_{n,0} = T_{p_\theta^S}(R) = T_{\Lambda_\theta}(SU(2,C)) \oplus T_{\Lambda_\theta}(SU(2,C)) \oplus \cdots \oplus T_{\Lambda_\theta}(SU(2,C))$. The computation is similar as that in the previous subsection, so we briefly go through the results. $w(t)$ is the tangent curve at identity of $SU(2,C)$ as before, we take a tangent curve to the point $p_\theta^S = (\Lambda_\theta,\, \Lambda_\theta,\, \cdots,\, \Lambda_\theta)$ to be $v(t) = (w(t)\cdot\Lambda_\theta,\, \Lambda_\theta,\, \cdots,\, \Lambda_\theta)$. Then, we have

$$\sigma_1^*(v(t)) = ((w(t)\cdot\Lambda_\theta\cdot w(t)^{-1}\cdot\Lambda_\theta^{-1})\cdot\Lambda_\theta,\, w(t)\cdot\Lambda_\theta,\, \Lambda_\theta,\, \cdots,\, \Lambda_\theta).$$

After taking derivative, we have

$$d\sigma^*_{1\,p_\theta^S}(w,\, 0,\cdots,\, 0) = (w - \Lambda_\theta w\Lambda_\theta^{-1},\, w,\, 0,\, \cdots,\, 0).$$

For the second component, taking the tangent curve to be

$$\widetilde{v}(t) = (\Lambda_\theta,\, w(t)\Lambda_\theta,\, \Lambda_\theta,\, \cdots,\, \Lambda_\theta),$$

we then have

$$\sigma_1^*\widetilde{v}(t) = (\Lambda_\theta w(t)\Lambda_\theta^{-1}\cdot\Lambda_\theta,\, \Lambda_\theta,\, \cdots,\, \Lambda_\theta).$$

Taking derivative, we have

$$d\sigma^*_{1\,p_\theta^S}(0,\, w,\cdots,\, 0) = (\Lambda_\theta w\Lambda_\theta^{-1},\, 0,\, \cdots,\, 0).$$

For any vector $X \in \mathcal{G}_1$, since $Ad(\Lambda_\theta)(X) = X$, the matrix representation for $\overline{\sigma}_1$ on the real part of $V_{n,0}$ is a permutation matrix. For any element $\begin{pmatrix} 0 & \zeta \\ -\overline{\zeta} & 0 \end{pmatrix}$ in the complex part $\mathcal{G}_2$ of the Lie algebra $\mathcal{G}$, we have, $Ad(\Lambda_\theta)\begin{pmatrix} 0 & \zeta \\ -\overline{\zeta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \zeta e^{2i\theta} \\ -\overline{\zeta}e^{-2i\theta} & 0 \end{pmatrix}$. That is, $\zeta \longmapsto \zeta e^{2i\theta}$. Set $e^{i\theta} = \lambda$, for any vector X in the complex part $\mathcal{G}_2$, we have

$$\begin{aligned} \overline{\sigma}_1(X,\, 0,\cdots,\, 0) &= (X - \Lambda_\theta X\Lambda_\theta^{-1},\, X,\, 0,\, \cdots,\, 0) \\ &= ((1-\lambda^2)X,\, X,\, 0,\, \cdots,\, 0), \end{aligned}$$

and

$$\begin{aligned}\overline{\sigma}_1(0,\ X,\ 0,\ \cdots,\ 0) &= (\Lambda_\theta X \Lambda_\theta^{-1},\ 0,\ \cdots,\ 0)\\ &= (\lambda^2 X,\ 0,\ \cdots,\ 0).\end{aligned}$$

Therefore, the matrix representation for $\overline{\sigma}_1$ over the complex part of $V_{n,0}$ is

$$\begin{pmatrix} 1-\lambda^2 & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}.$$

Similarly, we can get the matrix representations for other generators $\overline{\sigma}_j$, $1 \leq j \leq n-1$, that is,

$$I_{j-1} \oplus \begin{pmatrix} 1-\lambda^2 & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-j-1}.$$

We denote these matrices by G_j, and this is another family of representation building blocks. If we set $\lambda^2 = t$, we get the Burau representation for the braid group B_n.

3.3 Representations with the spins of $b(S) = 1$

We consider the spins S which have the negative index $b(S) = 1$, for example, $S = (-1,+1,\cdots,+1)$. There are n different spins that each has one negative sign. As before, we denote the point p_θ in the maximal torus marked by a spin S by p_θ^S. Specifically, denote $q_k = (\Lambda_\theta,\ \cdots,\ \Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta)$, where $k = 1,\ 2,\ \cdots,\ n$. So, in this case, we have

$$V_{n-1,1} = \bigoplus_{b(S)=1} T_{p_\theta^S}(R) = \oplus_{k=1}^{n} T_{q^k}(R).$$

We still focus on computation of the matrix representation of $\sigma_1 \in B_n$ firstly.

We know that

$$\sigma_1^* q_1 = \sigma_1^*(\Lambda_\theta^{-1},\Lambda_\theta,\cdots,\Lambda_\theta) = (\Lambda_\theta,\Lambda_\theta^{-1},\Lambda_\theta,\cdots,\Lambda_\theta) = q_2,$$

$$\sigma_1^* q_2 = \sigma_1^*(\Lambda_\theta,\Lambda_\theta^{-1},\Lambda_\theta,\cdots,\Lambda_\theta) = q_1,$$

and

$$\sigma_1^* q_k = q_k, \quad \text{for } 3 \leq k \leq n.$$

Also, we know in this case $\overline{\sigma}_1$ is given by

$$\overline{\sigma}_1 = d{\sigma_1^*}_{q_1} \oplus d{\sigma_1^*}_{q_2} \oplus d{\sigma_1^*}_{q_3} \oplus \cdots \oplus d{\sigma_1^*}_{q_n}.$$

We need to compute three tangent maps, derivatives.

Consider $d{\sigma_1^*}_{q_1} : T_{q_1}R \longrightarrow T_{q_2}R$. Let $w(t)$ to be a tangent curve to the identity of $SU(2,C)$ as before. Define a tangent curve to the point q_1 as

$$\begin{aligned} v(t): \quad (-1,\ 1) \quad &\longrightarrow R \\ t \quad &\longmapsto (w(t)\Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta), \end{aligned}$$

then, the action of σ_1^* is given by

$$\begin{aligned} \sigma_1^* v(t) &= (w(t)\Lambda_\theta^{-1}\cdot\Lambda_\theta\cdot(w(t)\Lambda_\theta^{-1})^{-1},\ w(t)\Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta) \\ &= (w(t)\Lambda_\theta w(t)^{-1}\Lambda_\theta^{-1}\cdot\Lambda_\theta,\ w(t)\Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta), \end{aligned}$$

which is a tangent curve to the point q_2. Evaluating at $t = 0$,

$$\frac{d}{dt}(w(t)\Lambda_\theta w(t)^{-1}\Lambda_\theta^{-1})|_{t=0} = w - \Lambda_\theta w \Lambda_\theta^{-1},$$

so, we have tangent map,

$$d{\sigma_1^*}_{q_1}(w,\ 0,\ \cdots,\ 0) = (w - \Lambda_\theta w\Lambda_\theta^{-1},\ w,\ 0,\ \cdots,\ 0).$$

If a tangent curve to the point q_1 is defined to be

$$\begin{aligned} \widetilde{v}(t): \quad (-1,\ 1) \quad &\longrightarrow R \\ t \quad &\longmapsto (\Lambda_\theta^{-1},\ w(t)\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta), \end{aligned}$$

the action of σ_1^* will be

$$\sigma_1^* \widetilde{v}(t) = (\Lambda_\theta^{-1} w(t) \Lambda_\theta \cdot \Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta).$$

After taking derivative, we have

$$d\sigma^*_{1\ q_1}(0,\ w,\ 0,\ \cdots,\ 0) = (\Lambda_\theta^{-1} w \Lambda_\theta,\ 0,\ \cdots,\ 0).$$

It is easy to see that the matrix representation of the derivative over the real part $V_{n-1,1}$ is a permutation matrix. By using the actions of $Ad(\Lambda_\theta)$ and $Ad(\Lambda_\theta^{-1})$ in the previous two subsections, we easily get the matrix for the derivative $d\sigma^*_{1\ q_1}$ over the complex part, which is

$$\begin{pmatrix} 1-\lambda^2 & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = E_1.$$

Consider $d\sigma^*_{1\ q_2} :\ T_{q_2}R \longrightarrow T_{q_1}R$. Define a tangent curve to the point q_2 to be

$$\begin{aligned} v(t) :\ \ (-1,\ 1) &\longrightarrow R \\ t &\longmapsto (w(t)\Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta), \end{aligned}$$

then we have a tangent curve to the point q_1, which is

$$\begin{aligned} \sigma_1^* v(t) &= (w(t)\Lambda_\theta \cdot \Lambda_\theta^{-1} \cdot (w(t)\Lambda_\theta)^{-1},\ w(t)\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta) \\ &= (w(t)\Lambda_\theta^{-1} w(t)^{-1} \Lambda_\theta \cdot \Lambda_\theta^{-1},\ w(t)\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta). \end{aligned}$$

Once taking derivative, we get

$$d\sigma^*_{1\ q_2}(w,\ 0,\ \cdots,\ 0) = (w - \Lambda_\theta^{-1} w \Lambda_\theta,\ w,\ 0,\ \cdots,\ 0).$$

In order to compute the second component, define a tangent curve to the point q_2 to be

$$\begin{aligned} \widetilde{v}(t) :\ \ (-1,\ 1) &\longrightarrow R \\ t &\longmapsto (\Lambda_\theta,\ w(t)\Lambda_\theta^{-1},\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta), \end{aligned}$$

then the action of σ_1^* is

$$\sigma_1^* \widetilde{v}(t) = (\Lambda_\theta w(t) \Lambda_\theta^{-1} \cdot \Lambda_\theta^{-1},\ \Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta).$$

After taking derivative, we have

$$d\sigma^*_{1\ q_2}(0,\ w,\ 0,\ \cdots,\ 0) = (\Lambda_\theta w \Lambda_\theta^{-1},\ 0,\ \cdots,\ 0).$$

Therefore, we get the matrix representation of the derivative $d\sigma^*_{1\ q_2}$ on the complex part, which is

$$\begin{pmatrix} 1-\lambda^{-2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = F_1.$$

Since, for $k \geq 3$, $\sigma_1^* q_k = q_k$, from subsection 3.2, we know that the matrix for $d\sigma^*_{1\ q_k}$ on the complex part is

$$\begin{pmatrix} 1-\lambda^{2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = G_1.$$

Therefore, we get the matrix representation for $\overline{\sigma}_1$ as follows

$$\begin{pmatrix} 0 & F_1 \\ E_1 & 0 \end{pmatrix} \oplus G_1 \oplus \cdots \oplus G_1 = M_{\sigma_1}.$$

This is $n^2 \times n^2$ matrix, or $n \times n$ block matrix.

Now we need to consider $\sigma_2 \in B_n$. It is clear that, $\sigma_2^* q_1 = q_1$, $\sigma_2^* q_2 = q_3$, $\sigma_2^* q_3 = q_2$ and $\sigma_2^* q_k = q_k$ for $k \geq 4$. For $d\sigma^*_{2\ q_1} : T_{q_1} R \longrightarrow T_{q_1} R$, after a similar computation, we get the corresponding matrix G_2 given by

$$I_1 \oplus \begin{pmatrix} 1-\lambda^{2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-3}.$$

For $d\sigma^*_{2\ q_2} : T_{q_2} R \longrightarrow T_{q_3} R$, the matrix is E_2 given by

$$I_1 \oplus \begin{pmatrix} 1-\lambda^{2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-3}.$$

For $d\sigma^*_{2\,q_3}:\ T_{q_3}R \longrightarrow T_{q_2}R$, the matrix is F_2 given by

$$I_1 \oplus \begin{pmatrix} 1-\lambda^{-2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-3}.$$

Therefore, we get the matrix for $\overline{\sigma}_2$ as follows:

$$G_2 \oplus \begin{pmatrix} 0 & F_2 \\ E_2 & 0 \end{pmatrix} \oplus G_2 \oplus \cdots \oplus G_2 = M_{\sigma_2}.$$

Denote

$$G_k = I_{k-1} \oplus \begin{pmatrix} 1-\lambda^{2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-k-1},$$

$$F_k = I_{k-1} \oplus \begin{pmatrix} 1-\lambda^{-2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-k-1},$$

and

$$E_k = I_{k-1} \oplus \begin{pmatrix} 1-\lambda^{2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-k-1}.$$

The first one is a Burau block in our context, the other two are new blocks. They are building blocks for our spin representations for the spins with $a(S) = n-1$. Using similar computations, we can get the matrix for the generator $\sigma_k \in B_n$, which is

$$M_{\sigma_k} = G_k \oplus \cdots G_k \oplus \begin{pmatrix} 0 & F_k \\ E_k & 0 \end{pmatrix} \oplus G_k \oplus \cdots \oplus G_k,$$

the first $k-1$ blocks and the last $n-k-1$ blocks are all G_k.

We here give a directly verification of the matrix representations for spins with the positive index $a(S) = n-1$, and this shows that these representations don't depend on the way we get them.

Lemma 3.1. *For the families of block matrices E_k, F_k and G_k, $1 \leq k \leq n$, there exist the following identities:*

$$F_k F_{k+1} G_k = G_{k+1} F_k F_{k+1},$$

$$E_k G_{k+1} F_k = F_{k+1} G_k E_{k+1},$$

$$G_k E_{k+1} E_k = E_{k+1} E_k G_{k+1},$$

and for $|i-j| \geq 2$,

$$F_i G_j = G_j F_i,\ E_i G_j = G_j E_i,\ G_i G_j = G_j G_i.$$

Proof. For simplicity, we consider $k = 1$ for the identities those are involved to three matrices.

$$\begin{aligned}
F_1 F_2 G_1 = \quad & (\begin{pmatrix} 1-\lambda^{-2} & \lambda^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}) \cdot (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\lambda^{-2} & \lambda^2 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_{n-3}) \cdot \\
& \cdot(\begin{pmatrix} 1-\lambda^2 & \lambda^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}) \\
& = (\begin{pmatrix} 1-\lambda^{-2} & \lambda^2-1 & \lambda^4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_{n-3}) \cdot (\begin{pmatrix} 1-\lambda^2 & \lambda^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}) \\
& = \begin{pmatrix} 1-\lambda^{-2} & \lambda^2-1 & \lambda^4 \\ 1-\lambda^2 & \lambda^2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus I_{n-3}.
\end{aligned}$$

And,

$$G_2F_1F_2 = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\lambda^2 & \lambda^2 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_{n-3}\right) \cdot \left(\begin{pmatrix} 1-\lambda^{-2} & \lambda^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}\right) \cdot$$

$$\cdot\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\lambda^{-2} & \lambda^2 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_{n-3}\right)$$

$$= \begin{pmatrix} 1-\lambda^{-2} & \lambda^2-1 & \lambda^4 \\ 1-\lambda^2 & \lambda^2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus I_{n-3}.$$

So, we get the identity $F_1F_2G_1 = G_2F_1F_2$. Similar computations show $E_1G_2F_1 = F_2G_1E_2$ and $G_1E_2E_1 = E_2E_1G_2$.

As to the commutativity identities, they are easy to see since non-identity blocks have never come together when taking multiplication. For example,

$$F_1G_3 = \left(\begin{pmatrix} 1-\lambda^{-2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_2 \oplus I_{n-4}\right) \cdot \left(I_2 \oplus \begin{pmatrix} 1-\lambda^2 & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-4}\right) = G_3E_1.$$

□

Theorem 3.2. *For* $k = 1,2,\cdots,n-2$,

$$M_{\sigma_k}M_{\sigma_{k+1}}M_{\sigma_k} = M_{\sigma_{k+1}}M_{\sigma_k}M_{\sigma_{k+1}},$$

and for $|i-j| \geq 2$,

$$M_{\sigma_i}M_{\sigma_j} = M_{\sigma_j}M_{\sigma_i}.$$

Proof. It is enough to check $M_{\sigma_1}M_{\sigma_2}M_{\sigma_1} = M_{\sigma_2}M_{\sigma_1}M_{\sigma_2}$ and $M_{\sigma_1}M_{\sigma_3} =$

$M_{\sigma_3}M_{\sigma_1}$. We see that

$$\begin{aligned} M_{\sigma_1}M_{\sigma_2}M_{\sigma_1} &= (\begin{pmatrix} 0 & F_1 & 0 \\ E_1 & 0 & 0 \\ 0 & 0 & G_1 \end{pmatrix} \oplus G_1 \oplus \cdots G_1) \cdot (\begin{pmatrix} G_2 & 0 & 0 \\ 0 & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix} \oplus \\ & \oplus G_2 \oplus \cdots G_2) \cdot (\begin{pmatrix} 0 & F_1 & 0 \\ E_1 & 0 & 0 \\ 0 & 0 & G_1 \end{pmatrix} \oplus G_1 \oplus \cdots G_1). \end{aligned}$$

Since G_k is a Burau block, we have $G_1G_2G_1 = G_2G_1G_2$. We only need to check

$$\begin{pmatrix} 0 & F_1 & 0 \\ E_1 & 0 & 0 \\ 0 & 0 & G_1 \end{pmatrix} \begin{pmatrix} G_2 & 0 & 0 \\ 0 & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & F_1 & 0 \\ E_1 & 0 & 0 \\ 0 & 0 & G_1 \end{pmatrix}$$

$$= \begin{pmatrix} G_2 & 0 & 0 \\ 0 & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & F_1 & 0 \\ E_1 & 0 & 0 \\ 0 & 0 & G_1 \end{pmatrix} \begin{pmatrix} G_2 & 0 & 0 \\ 0 & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix}.$$

After a simply matrix calculation, it reduces to

$$\begin{pmatrix} 0 & 0 & F_1F_2G_1 \\ 0 & E_1G_2F_1 & 0 \\ G_1E_2E_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & G_2F_1F_2 \\ 0 & F_2G_1E_2 & 0 \\ E_2E_1G_2 & 0 & 0 \end{pmatrix}.$$

By the Lemma 3.1, this is an identity. We compute

$$M_{\sigma_1}M_{\sigma_3} = \begin{pmatrix} 0 & F_1G_3 \\ E_1G_3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & G_1F_3 \\ G_1E_3 & 0 \end{pmatrix} \oplus G_1G_3 \oplus \cdots G_1G_3;$$

and

$$M_{\sigma_3}M_{\sigma_1} = \begin{pmatrix} 0 & G_3F_1 \\ G_3E_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & F_3G_1 \\ E_3G_1 & 0 \end{pmatrix} \oplus G_3G_1 \oplus \cdots G_3G_1.$$

By the Lemma 3.1, we see $M_{\sigma_1}M_{\sigma_3} = M_{\sigma_3}M_{\sigma_1}$. $\square$

3.4 Representations with the spins of $b(S) = 2,\ b$

In order to show that four types of representation building blocks are needed for general spin representations, we now work on a case where each spin has the negative index $b(S) = 2$, for example $S = (-1,-1,+1,\cdots,+1)$. There are $\frac{n(n-1)}{2}$ such spins. For any fixed θ, there are $\frac{n(n-1)}{2}$ point in the maximal torus marked by these spins. For certainty, we denote them as follows. $q_{1\,2} = (\Lambda_\theta^{-1},\ \Lambda_\theta^{-1},\ \Lambda_\theta,\cdots,\Lambda_\theta)$, $q_{1\,3} = (\Lambda_\theta^{-1},\ \Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta,\cdots,\Lambda_\theta)$, $\cdots$, $q_{1\,n} = (\Lambda_\theta^{-1},\ \Lambda_\theta,\ \Lambda_\theta,\cdots,\Lambda_\theta,\ \Lambda_\theta^{-1})$; $q_{2\,3} = (\Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta^{-1},\ \Lambda_\theta,\cdots,\Lambda_\theta)$, $q_{2\,4} = (\Lambda_\theta, \Lambda_\theta^{-1}, \Lambda_\theta, \Lambda_\theta^{-1}, \Lambda_\theta,\cdots,\Lambda_\theta)$, $\cdots$, $q_{2\,n} = (\Lambda_\theta, \Lambda_\theta^{-1}, \Lambda_\theta,\cdots,\Lambda_\theta, \Lambda_\theta^{-1})$; $\cdots\cdots$, $q_{n-2\,n-1} = (\Lambda_\theta,\cdots,\Lambda_\theta, \Lambda_\theta^{-1}, \Lambda_\theta^{-1}, \Lambda_\theta)$, $q_{n-2\,n} = (\Lambda_\theta,\cdots,\Lambda_\theta, \Lambda_\theta^{-1}, \Lambda_\theta\Lambda_\theta^{-1})$; and $q_{n-1\,n} = (\Lambda_\theta,\cdots,\Lambda_\theta,\ \Lambda_\theta^{-1},\ \Lambda_\theta^{-1})$. We order those tangent spaces lexicographically, then we have

$$V_{n-2,\,2} = T_{q_{1\,2}}R \oplus T_{q_{1\,3}}R \oplus \cdots \oplus T_{q_{1\,n}} \oplus \cdots\cdots \oplus T_{q_{n-2\,n}} \oplus T_{q_{n-1\,n}}.$$

For the generator of the braid group σ_1, its image is given by

$$\overline{\sigma}_1 = d\sigma_1^*{}_{q_{12}} \oplus d\sigma_1^*{}_{q_{13}} \oplus \cdots \oplus d\sigma_1^*{}_{q_{1n}} \oplus \cdots \oplus d\sigma_1^*{}_{q_{n-1\,n}}.$$

We see $\sigma_1^*(q_{1\,2}) = q_{1\,2}$, as before, we can linearize it, and compute its matrix representation at the point $q_{1\,2}$ over the complex part of $V_{n-2,\,2}$, which is

$$\begin{pmatrix} 1-\lambda^{-2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = D_1.$$

For $\sigma_1^*(q_{1\,3}) = q_{2\,3}$, we use the same procedure, and get its matrix representation at the point $q_{1\,3}$, which is

$$\begin{pmatrix} 1-\lambda^{2} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = E_1.$$

Actually, since $\sigma_1^*(q_{1\,k}) = q_{2\,k}$ for $k = 3, 4, \cdots, n$, their linearizations are all the same, and their matrix representations are all E_1, but at different marked points. Since $\sigma_1^*(q_{2\,k}) = q_{1\,k}$ for $k = 3, 4, \cdots, n$, their matrix representations are

$$\begin{pmatrix} 1-\lambda^{-2} & \lambda^2 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2} = F_1.$$

For the point $q_{i\,j}$, where $i \geq 3$ and $j \geq 4$, the negative sign of spins does not affect the tangent map $d{\sigma_1^*}_{q_{ij}}$, so its matrix representation is a Burau block G_1. Let's sum up, the matrix representation of $\overline{\sigma}_1$ is given by

$$D_1 \oplus \begin{pmatrix} 0 & \widetilde{F}_1 \\ \widetilde{E}_1 & 0 \end{pmatrix} \oplus I_{(n-2)(n-3)/2} \otimes G_1,$$

where $\widetilde{F}_1 = I_{n-2} \otimes F_1$ and $\widetilde{E}_1 = I_{n-2} \otimes E_1$.

For the generator $\sigma_k \in B_n$, we have the following proposition to state how many block matrices in its representation. The proof is directly from the representation matrix $\overline{\sigma}_1$ above.

Proposition 3.1. *The representations with spins of $b(S) = 2$ for $\sigma_k \in B_n$ have one block of D_k, $n-2$ blocks of F_k, $n-2$ blocks of E_k, and $(n-2)(n-3)/2$ blocks of G_k.*

For spins with negative index $b(S) = b \geq 2$, the matrix representations of the generator σ_k have the following proposition.

Proposition 3.2. *For any matrix representation of the generator σ_k with spins of $b(S) = b$, it is a $\binom{n}{b} \times \binom{n}{b}$ block matrix. There are $\binom{n-2}{b-2}$ blocks of D_k, $\binom{n-2}{b-1}$ blocks of E_k, $\binom{n-2}{b-1}$ blocks of F_k, and $\binom{n-2}{n-b-2}$ blocks of G_k in this block matrix.*

Proof. Consider the first two components of the maximal torus $S^1 \times S^1 \times \cdots \times S^1$ with n copies, for given negative index b, there are $\binom{n-2}{b-2}$ possibility where both of them have negative signs, $\binom{n-2}{n-b-2}$ possibility where both of them have positive signs, $\binom{n-2}{b-1}$ possibility where one of them has negative sign and the other has positive sign. For each of these possible position, we compute the derivative map of σ_1^*, and count them as blocks in the representation matrix of $\overline{\sigma}_1$. For σ_k, we just consider k and $k+1$ components of the maximal torus, and will get the numbers. It is clear that the representation matrix of $\overline{\sigma}_k$ has size $\binom{n}{b} \times \binom{n}{b}$ since there are $\binom{n}{b}$ different points on the maximal torus. $\square$

3.5 Spins and the Weyl group

Spins actually come up naturally with the action of the Weyl group. Consider $SU(2,C)$, its maximal torus is S^1. The Weyl group of $SU(2,C)$ is the quotient group, the normalizer $N(S^1,\, SU(2,C))$ of S^1 over S^1 itself. Here, $N(S^1,\, SU(2,C)) = \{g \in SU(2,C) : gS^1g^{-1} = S^1\} = \{\begin{pmatrix} 0 & b \\ -\overline{b} & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} : a\overline{a} = b\overline{b} = 1\}$. So, the Weyl group is $W(SU(2,C)) = \{S^1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S^1\} = Z_2$. The $W(SU(2,C))$ acts on the maximal torus S^1, also on the Lie algebra $\mathcal{G}$. For example, check the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S^1$ on Λ_θ:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S^1.\Lambda_\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \Lambda_\theta^{-1},$$

so, we get a negative sign. Also it is easy to see that $S^1.\Lambda_\theta = \Lambda_\theta$. For simplicity, denote elements of $W(SU(2,C))$ as $S^1 = \overline{0}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S^1 = \overline{1}$, then $Z_2 = \{\overline{0}, \overline{1}\}$. Now we look at $R = SU(2,C)^n$. The Weyl group $W(R)$ of R is Z_2^n. For any spin $S = (s_1,\ s_2,\ \cdots,\ s_n)$, and a point $p_\theta = (\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta)$ on the maximal torus $S^1 \times S^1 \times \cdots \times S^1$, as before $p_\theta^S = (\Lambda_\theta^{s_1},\ \Lambda_\theta^{s_2},\ \cdots,\ \Lambda_\theta^{s_n})$. We choose an element of $W(R)$ for the spin S, denote it as $\pi(S) = (\pi_1, \pi_2, \cdots, \pi_n)$, according to the following rules:

$$\pi_j = \begin{cases} \overline{0}, & \text{if } s_j = 1 \\ \overline{1}, & \text{if } s_j = -1. \end{cases}$$

It is easy to check that this is a 1-1 and onto map between the Wely group $W(R)$ and the set of all spins with n components. From the action of the Weyl group $W(SU(2,C))$ on S^1, we can easy get the following property, which tells us spins naturally come from the action of the Weyl group.

Proposition 3.3.

$$p_\theta^S = \pi(S).p_\theta.$$

Since the Weyl group $W(SU(2,C))$ can also act on the Lie algebra $\mathcal{G}$, the different types of block matrices in representations are naturally coming from this action. It is enough to check with σ_1. We write it as a proposition.

Proposition 3.4. *Let* $\pi = (\pi_1, \pi_2, \cdots \pi_n,) \in Z^n$ *and* $p_\theta = (\Lambda_\theta, \Lambda_\theta, \cdots, \Lambda_\theta)$, *for any element* $w \in \mathcal{G}$,

$$d\sigma^*_{1\,\pi.p_\theta}(w, 0, \cdots, 0) = (w - \pi_2.\Lambda_\theta \cdot w \cdot (\pi_2.\Lambda_\theta)^{-1}, w, 0, \cdots, 0),$$

and

$$d\sigma^*_{1\,\pi.p_\theta}(0, w, 0, \cdots, 0) = (\pi_1.\Lambda_\theta \cdot w \cdot (\pi_1.\Lambda_\theta)^{-1}, 0, \cdots, 0).$$

The proof is a similar calculation as we did in previous subsections.

We can see that, when $\pi_1 = \overline{0}$ and $\pi_2 = \overline{0}$, we get a Burau block G; when $\pi_1 = \overline{1}$ and $\pi_2 = \overline{1}$, we get a block matrix D; when $\pi_1 = \overline{1}$ and $\pi_2 = \overline{0}$, we get a block matrix F; $\pi_1 = \overline{0}$ and $\pi_2 = \overline{1}$, we get a block matrix E. There are only four different elements in the first two components in the Weyl group Z^n, so we only can have four types of representation building blocks.

4 Multi-parameter Burau representations

We now mark each component of the maximal torus $S^1 \times S^1 \times \cdots S^1$ of $SU(2,C)^n = R$ with different parameters, or colors, say, $t = (t_1,t_2,\cdots,t_n)$, where $t \in \Re$ the real number field. Still using $\Lambda_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ and $p_\theta = (\Lambda_\theta,\ \Lambda_\theta,\ \cdots,\ \Lambda_\theta)$, then denote $p_\theta^t = (\Lambda_\theta^{t_1},\ \Lambda_\theta^{t_2},\ \cdots,\ \Lambda_\theta^{t_n})$. It is easy to see that $p_\theta^t \in SU(2,C)$. Let Σ_n be the symmetry group, for any fixed θ and t, there are $n!$ points on the maximal torus. So we define a linear space, which is a summation of all tangent spaces of R at these points and each of them is marked by its base points,

$$V_t = \bigoplus_{\tau \in \Sigma_n} T_{p_\theta^{\tau(t)}}(R),$$

where τ acts on t in the obvious way. For any braid $\beta \in B_n$, $\beta^* \in Diff(R)$, and

$$\begin{aligned}
\beta^*(p_\theta^t) &= \beta^*(\Lambda_\theta^{t_1},\ \Lambda_\theta^{t_2},\ \cdots,\ \Lambda_\theta^{t_n}) \\
&= (\Lambda_\theta^{t_{\tau_\beta(1)}},\ \Lambda_\theta^{t_{\tau_\beta(2)}},\ \cdots,\ \Lambda_\theta^{t_{\tau_\beta(n)}}) \\
&=: p_\theta^{\tau_\beta(t)}.
\end{aligned}$$

Define a map from B_n to $GL(V_t)$, for each $\beta \in B_n$, define its image to be

$$\overline{\beta} = \bigoplus_{\tau \in \Sigma_n} d\beta^*_{p_\theta^{\tau(t)}}.$$

So,

$$\overline{\beta} : \; V_t \longrightarrow V_t,$$

if $X \in T_{p_\theta^t}(R) \subset V_t$, then $\overline{\beta}(X) = d\beta^*_{p_\theta^t}(X) \in T_{p_\theta^{\tau_\beta(t)}}(R) \subset V_t$. For any two braids β_1 and β_2 in B_n, it is obvious that

$$d\beta^*_{p_\theta^{\tau(t)}} \cdot d\beta^*_{p_\theta^{\tau(t')}} = \begin{cases} d(\beta_1\beta_2)^*_{p_\theta^{t'}}, & \text{if } t = \tau_{\beta_2}(t'), \\ 0, & \text{otherwise.} \end{cases}$$

Now we have a statement that the map we define is a group homomorphism.

Theorem 4.1. *The map from B_n to $GL(V_t)$ given by $\overline{\beta}$ for each $\beta \in B_n$ is a group homomorphism. That is $\overline{\beta_1\beta_2} = \overline{\beta}_1\overline{\beta}_2$.*

Proof. The proof is a simple check.

$$\begin{aligned} \overline{\beta}_1\overline{\beta}_2 &= \bigoplus_{\tau \in \Sigma_n} d\beta^*_{1\, p_\theta^{\tau(t)}} \cdot \bigoplus_{\tau \in \Sigma_n} d\beta^*_{2\, p_\theta^{\tau(t)}} \\ &= \bigoplus_{\tau \in \Sigma_n} d\beta^*_{1\, p_\theta^{\tau_{\beta_2}(\tau(t))}} \cdot d\beta^*_{2\, p_\theta^{\tau(t)}} \\ &= \overline{\beta_1\beta_2}. \end{aligned}$$

□

Let's compute the matrix representation for any given braid. Consider the generator σ_1. We know that $\overline{\sigma}_1 = \oplus_{\tau \in \Sigma_n} d\sigma^*_{1\, p_\theta^{\tau(t)}}$. Since $\sigma_1^*(\Lambda_\theta^{t_1}, \Lambda_\theta^{t_2}, \cdots, \Lambda_\theta^{t_n}) = (\Lambda_\theta^{t_2}, \Lambda_\theta^{t_1}, \Lambda_\theta^{t_3}, \cdots, \Lambda_\theta^{t_n})$, for a tangent curve, we need to put its image at base point. That is,

$$\begin{aligned} \sigma_1^*(w(s)\Lambda_\theta^{t_1}, \Lambda_\theta^{t_2}, \cdots, \Lambda_\theta^{t_n}) &= (w(s)\Lambda_\theta^{t_1} \cdot \Lambda_\theta^{t_2} \cdot (w(s)\Lambda_\theta^{t_1})^{-1}, w(s)\Lambda_\theta^{t_1}, \Lambda_\theta^{t_3}, \cdots, \Lambda_\theta^{t_n}) \\ &= (w(s)\Lambda_\theta^{t_2} w(s)^{-1}\Lambda_\theta^{-t_2} \cdot \Lambda_\theta^{t_2}, w(s)\Lambda_\theta^{t_1}, \Lambda_\theta^{t_3}, \cdots, \Lambda_\theta^{t_n}), \end{aligned}$$

where $w(s)$ is a tangent curve to the identity of $SU(2,C)$ defined as before. Taking derivative, $\frac{d}{ds}(w(s)\Lambda_\theta^{t_2}w(s)^{-1}\Lambda_\theta^{-t_2})|_{s=0} = w - \Lambda_\theta^{t_2}w\Lambda_\theta^{-t_2}$, so we have

$$d\sigma^*_{1\,p^t_\theta}(w,\ 0,\cdots,\ 0) = (w - \Lambda_\theta^{t_2}w\Lambda_\theta^{-t_2},\ w,\ 0,\ \cdots,\ 0).$$

To compute the second component of the image under the tangent map $d\sigma^*_{1\,p^t_\theta}$, similarly, consider the tangent curve $(\Lambda_\theta^{t_1},w(s)\Lambda_\theta^{t_2},\Lambda_\theta^{t_3},\cdots,\Lambda_\theta^{t_n})$, and we get

$$d\sigma^*_{1\,p^t_\theta}(0,\ w,\ 0,\ \cdots,\ 0) = (\Lambda_\theta^{t_1}w\Lambda_\theta^{-t_1},\ 0,\ \cdots,\ 0).$$

If w is in the complex part of the Lie algebra $\mathcal{G}$, we see its matrix representation is

$$\begin{pmatrix} 1-\lambda^{t_2} & \lambda^{t_1} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2},$$

where we set $e^{2i\theta} = \lambda$. We similarly compute the matrix representations of other tangent maps $d\sigma^*_{1\,p^{\tau(t)}_\theta}$, denote

$$M_1(t_i,t_j) = \begin{pmatrix} 1-\lambda^{t_j} & \lambda^{t_i} \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}.$$

Then we have a matrix representation for $\overline{\sigma}_1$, which is

$$M_{\sigma_1} = \bigoplus_{i\neq j} M_1(t_i,t_j).$$

Generally, for σ_k, $1 \le k \le n-1$, we have

$$M_k(t_i,t_j) = I_{k-1} \oplus \begin{pmatrix} 1-\lambda^{t_j} & \lambda^{t_i} \\ 1 & 0 \end{pmatrix} \oplus I_{n-k-1},$$

and the matrix representation for $\overline{\sigma}_k$ is

$$M_{\sigma_k} = \bigoplus_{i\neq j} M_k(t_i,t_j).$$

Remark 1. *Eigenvalues of representations*

For spin representations, when the positive index $a(S) = n$, which is the Burau representation, it is easy to show that the matrix representation of each generator of B_n satisfies the characteristic polynomial, here it is also minimal polynomial, $x^2 = (1-\lambda^2)x + \lambda^2$. If $b(S) = n$, they satisfy $x^2 = (1-\lambda^{-2})x + \lambda^{-2}$. Both of them have two distinct eigenvalues. Jones studied all representations $\rho : B_n \longrightarrow GL_n(C)$ which have at most two distinct eigenvalues [3]. His Heche algebra is a complex algebra defined by generators $g_1,\ g_2,\ \cdots, g_n$ with defining relations,

$$g_i g_j = g_j g_i \text{ if } |i-j| \geq 2,\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ g^2 = (1-t)g_i + t.$$

The last quadratic equation is a version of the minimal polynomial for Burau. By using Cayley-Hamilton theorem with Frobenius theorem, we get the minimal polynomial for spin representations with the negative index $b(S) = 1$, which is

$$(x^2 + (\lambda^2 - 1)x - \lambda^2)(x^2 - 1)^2.$$

It has a degree of six and four distinct eigenvalues. For spin representations with the negative index $b(S) \geq 2$, the minimal polynomial is

$$(x^2 + (\lambda^2 - 1)x - \lambda^2)(x^2 + (\lambda^{-2} - 1)x - \lambda^{-2})(x^2 - 1)^2,$$

it has a degree of eight and six distinct eigenvalues. It is little bit difficult to compute the minimal polynomial of the multi-parameter Burau representations. But it is sure that the minimal polynomial of the multi-parameter Burau representations will have at least 4 distinct eigenvalues, and a degree of at least six. We may conclude that these representations can not arise from any Hecke algebra, and may hope that they can encode more information about the braid group.

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