A brief introduction to evolution algebras

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Abstract

A brief introduction to evolution algebras is presented. Evolution algebras are motivated by genetics and Markov Chains. They are non-associative and commutative. Evolution algebras possess distinct concepts such as algebraic persistency, algebraic transiency and algebraic periodicity. These notions lead to hierarchical structures for evolution algebras. Algebraically, a hierarchical structure is given by a sequence of semidirect-sum decompositions of an evolution algebra; dynamically, a hierarchical structure displays directions of the flows induced by an evolution algebra. The hierarchical structure is preserved under evolutionary isomorphisms. Skeleton classification and skeleton-shape classification of evolution algebras are given. Each evolution algebra is evolutionary homomorphic to its skeleton.

1 Introduction

In this article, we briefly introduce a new type of algebra, that we call evolution algebra. It is motivated by evolution laws of genetics [1][2]. We

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view the generators of the algebra as alleles (or organelles or cells), then define the multiplication of two "alleles" G_i and G_j by $G_i \cdot G_j = 0$ if $i \neq j$. However, $G_i \cdot G_i$ is viewed as "self-reproduction", so that $G_i \cdot G_i = \sum_j p_{ij} G_j$, where the summation is taken over all generators G_j. It seems obvious that this type of algebra is non-associative, but commutative if we keep in mind that reproduction in genetics is represented by multiplication in algebras. In history, many mathematicians studied algebras corresponding to Mendelian genetics [3][4][5][6][7][8]. Evolution algebras are corresponding to non-Mendelian genetics [2][13]. When the p_{ij}'s form Markovian transition probabilities, the properties of the algebra are associated with properties of Markov chains. Markov chains allow us to develop the algebras at deeper hierarchical levels than standard algebras. After we introduce several new algebraic concepts, particularly, algebraic persistency, algebraic transiency, algebraic periodicity, we establish the hierarchical structure for evolution algebras. The evolution algebras enable us to derive new results on Markov chains at the same time. When we apply the algebras back to non-Mendelian genetics, we can explain many puzzling features of organelle heredity[14]. We will give basic definitions and properties in this paper, and most of them we don't give detailed proof. A detailed study of evolution algebras can be found in my book [10].

2 Definitions and Basic Properties

In this section, we introduce the algebraic foundation of evolution algebras. The basic reference are [11], [12], [9] and [10].

2.1 Basic definitions

We define evolution algebras in terms of generators and defining relations. The method of generators and relations is similar to the axiomatic method, where the role of axioms is played by the defining relations.

Definition 2.1. *Let* $X = \{x_1, x_2, \cdots, x_v, \cdots\}$ *be the set of generators and* $R = \{f_l = x_l^2 + \sum_{k=1}^{v} a_{l,k}x_k,\ f_{ij} = x_i x_j \mid a_{l,k} \in K, i \neq j, l, i, j = 1, 2, \cdots, v, \cdots\}$ *be the set of defining relations, where* K *is a field, the evolution algebra is then defined by*

$$E(X) = \left\langle x_1, \cdots, x_v, \cdots \mid\ x_l^2 + \sum_{k=1}^{v} a_{l,k}x_k,\ x_i x_j, i \neq j; i, j, l \in \Lambda \right\rangle$$

where Λ *is the index set,* $\Lambda = \{1, 2, \cdots, v, \cdots\}$.

An algebra is real, if $K = \mathbb{R}$. *An evolution algebra is nonnegative if it is real and all structural constants* a_{ij} *are nonnegative.*

we have the following theorems.

Theorem 2.1. *(In page 19 of [10]) If the set of generators* X *is finite, then the evolution algebra* $E(X)$ *is finite dimensional. Moreover the set of generators* X *can serve as a basis of the algebra* $E(X)$. X *is called a natural basis of* $E(X)$.

Theorem 2.2. *(In page 20 of [10])*

(1) *Evolution algebras are not associative, in general.*

(2) *Evolution algebras are commutative, and hence flexible.*

(3) *Evolution algebra are not power-associative, in general.*

(4) *The direct sum of evolution algebras is an evolution algebra.*

(5) *The Kronecker product of evolution algebras is an evolution algebra.*

Definition 2.2. **(1)** *Let A be an evolution algebra, and B be a subspace of A. If B has a natural basis $\{g_i \mid i \in \Lambda_0\}$, which can be extended to a natural basis $\{g_j \mid j \in \Lambda\}$ of A, B is called an evolution subalgebra, where Λ_0 is a subset of Λ.*

(2) *Let A be a commutative algebra, we define principal powers of a respect to $b \in A$ as follows*

$$a^1 \odot b = ab, \quad a^2 \odot b = a(a^1 \odot b), \quad \cdots\cdots, \quad a^n \odot b = a(a^{n-1} \odot a).$$

And plenary powers of $a \in A$ as follows

$$\begin{aligned} a^{[1]} = a^{(2)} &= a \cdot a, \qquad a^{[2]} = a^{(4)} = a^{[1]} \cdot a^{[1]}, \quad \cdots\cdots, \\ a^{[n]} &= a^{(2^n)} = a^{[n-1]} \cdot a^{[n-1]}. \end{aligned}$$

For convenience, we denote $a^{[0]} = a$. Then, we have a property

$$\left(a^{[n]}\right)^{[m]} = a^{[n+m]},$$

where n and m are nonnegative integers.

(3) *We say an evolution algebra E is connected if E can not be decomposed into a direct sum of two proper evolution subalgebras.*

(4) *An evolution algebra E is simple if it has no proper evolution subalgebra.*

(5) *Let A and B be evolution algebras, a linear homomorphism f from A to B is called evolution homomorphism if f is an algebraic map and for a natural basis X of A, $f(X)$ spans an evolution subalgebra of B.*

2.2 Occurrence relation

Let E be an evolution algebra with the generator set $\{g_1, g_2, \cdots, g_v\}$. We say g_i occurs in $x \in E$ if the coefficient $\alpha_i \in K$ is nonzero in $x = \sum_{j=1}^{v} \alpha_j g_j$, denote by $g_i \prec x$.

It is easy to see that if $g_i \prec g_j^{[n]}$, then $\langle g_i \rangle \subseteq \langle g_j \rangle$, where $\langle x \rangle$ denotes the evolution subalgebra generated by x.

Proposition 2.1. *(In page 26 of [10]) Let E be a nonnegative evolution algebra. When $g_i \prec g_j^{[n]}$ and $g_j \prec g_k^{[m]}$, then $g_i \prec g_j^{n+m}$.*

We then have a type of partial order relation among the generator set of an evolution algebra E. Let g_i and g_j be two generators of E, if g_i occurs in a plenary power of g_j, define $g_i \leq g_j$. Then this is a partial order in the following sense.

(1) $g_i \leq g_i$ for any generator of E.

(2) If $g_i \leq g_j$ and $g_j \leq g_i$, then we say g_i and g_j intercommunicate. Generally, g_i and g_j are not necessarily the same, the evolution subalgebra generated by g_i and the one by g_j are the same, $\langle g_i \rangle = \langle g_j \rangle$.

(3) If $g_i \leq g_j$ and $g_j \leq g_k$, then $g_i \leq g_k$.

2.3 Evolution operators and non-associative Banach algebras

Definition 2.3. *Let E be an evolution algebra with a generator set $\{g_i \mid i \in \Lambda\}$. We define a $K-$linear map $\bigwedge$ as*

$$\begin{aligned} \bigwedge: \quad & E \longrightarrow E \\ & g_i \mapsto g_i^2 \quad \forall\ i \in \Lambda \end{aligned}$$

then linearly extend onto E.

Alternatively, by using a formal element $\Theta = \sum_{i\in\Lambda} g_i$, the evolution operator is defined by

$$\bigwedge(x) = \Theta \cdot x = (\sum_{i\in\Lambda} g_i) \cdot x,$$

for $x \in E$.

Theorem 2.3. *(In page 30 of [10]) If A is an evolution subalgebra of an evolution algebra E, then the evolution operator L of E leaves A invariant, namely, $\bigwedge(A) \subseteq A$.*

When E is a real evolution algebra, we can equip it with the usual L_1 norm, i.e., $\| \sum \alpha_i g_i \| = \sum \mid \alpha_i \mid$. Then E becomes a complete linear space with respect to the metric $\rho(x, y) = \| x - y \|$. In other words, E is a Banach space. For finite dimensional evolution algebra E, it is a non-associative Banach algebra.

When the dimension $dim(E)$ is finite, all linear operators defined on E are continuous. Particularly, every left translation by z, defined by $L_z(x) = zx$, is continuous. Although the composition of two left translations is no longer a left translation due to the lack of associativity, all left translations generate a subalgebra of $Hom(E, E)$, called the operator algebra of left multiplication of the algebra E, denoted by $L(E)$. If $dimE^2 \neq 1$, we have $dim(L(E)) > dim(E)$.

The subalgebra of $Hom(E, E)$ generated by all left and right multiplication operators is called the multiplication algebra of E, denote $M(E)$. The centroid centralizes the multiplication algebra $M(E)$, denote $\Gamma(E)$.

Theorem 2.4. *(In page 36 of [10]) Any evolution algebra is centroidal, i.e., $\Gamma(E) = K$.*

3 Periodicity and algebraic persistency

In this section, we introduce the periodicity for each generator of any evolution algebra. It turns out all generators of a simple evolution algebra have the same periodicity. We also introduce the algebraic persistency or the algebraic transiency for generators of any evolution algebra. They are basic concepts in the study of evolution in algebras.

3.1 Periodicity of generators in an evolution algebra

Definition 3.1. *Let g_j be a generator of an evolution algebra E, the period $d(g_j)$ of g_j is defined to be the greatest common divisor of the set $\left\{\log_2 m \mid g_j \prec \left(g_j^{(m)}\right)\right\}$, where power $g_j^{(m)}$ is some $k-th$ plenary power, $2^k = m$. That is*

$$d(g_j) = g.c.d. \left\{\log_2 m \mid g_j \prec \left(g_j^{(m)}\right)\right\}.$$

If this set is empty, we set $d(g_j) = \infty$; $d(g_j) = 1$, we say g_j is aperiodic.

Proposition 3.1. *(In page 40 of [10]) Generator g_j has a period d if and only if d is the greatest common divisor of the set $\{n \mid \rho_j\ \Theta^n \odot g_j) \neq 0\}$. That is*

$$d(g_j) = g.c.d.\{n \mid \rho_j\ \Theta^n \odot g_j) \neq 0\},$$

where ρ_i is a projection map of E, which maps an element of E to its g_i component.

Theorem 3.1. *All generators have the same period in a nonnegative simple evolution algebra.*

For the purpose of illustration, we give a simple proof here.

Proof. Let g_i and g_j be two generators in a nonnegative simple evolution algebra E. The periods of g_i and g_j are d_i and d_j respectively. Since g_i must occur in a plenary power of g_j, say $g_i \prec e_j^{[n]}$ and g_j must occur in a plenary power of g_i, say $g_j \prec g_i^{[m]}$, we have $g_i \prec g_i^{[n+m]}$ and $g_j \prec e_j^{[n+m]}$. Then $d_i \mid n+m$, and $d_j \mid n+m$. Since $g_j \prec g_j^{[d_j]}$, so $g_i \prec g_j^{[d_j+n]}$ and $g_i \prec g_i^{[d_j+n+m]}$, then $d_i \mid d_j+n+m$. Therefore $d_i \mid d_j$. Similarly, we have $d_j \mid d_i$. Thus, we get $d_i = d_j$. □

3.2 Algebraic persistency and algebraic transiency

Generator g_j is said to be algebraically persistent if the evolution subalgebra $\langle g_j \rangle$, generated by g_j, is a simple subalgebra, and g_i is algebraically transient if the subalgebra $\langle g_i \rangle$ is not simple. Then, it is obvious that every generator in a nonnegative simple evolution algebra is algebraically persistent, since each generator can generate the algebra that is simple. We know that if $x \leq y$ and $y \leq x$, the evolution subalgebra generated by x is the same as the one generated by y. Moreover, we have the following theorem.

Theorem 3.2. *Let g_i and g_j be generators of an evolution algebra E. If $g_i \leq g_j$ and $g_j \leq g_i$ and both are algebraically persistent, then they belongs to the same simple evolution subalgebra of E.*

Proof. Since $g_i \leq g_j$ and $g_j \leq g_i$, g_i occurs in $\langle g_j \rangle$ and g_j occurs in $\langle g_i \rangle$. Then, there are some powers of g_i, denoted by $P(g_i)$ and some powers of g_j, denoted by $Q(g_j)$, such that:

$$\begin{aligned} P(g_i) &= a g_j + u \qquad a \neq 0, \\ Q(g_j) &= b g_i + v \qquad b \neq 0. \end{aligned}$$

Since subalgebras are also ideals in an evolution algebra, we have

$$\begin{aligned} P(g_i)g_j &= ag_j^2 \in \langle g_i \rangle , \\ Q(g_j)g_i &= ag_i^2 \in \langle g_j \rangle . \end{aligned}$$

Therefore, $\langle g_i \rangle \cap \langle g_j \rangle \neq \{0\}$. Since $\langle g_i \rangle$ and $\langle g_j \rangle$ are both simple evolution subalgebras, then $\langle g_i \rangle = \langle g_j \rangle$. Thus, g_i and g_j belong to the same simple evolution subalgebra. □

For an evolution algebra, we can give certain conditions to specify whether it is simple or not by the following corollary.

Corollary 3.1. (1) *Let E be a connected evolution algebra, then E has a proper evolution subalgebra if and only if E has an algebraically transient generator.*

(2) *Let E be a connected evolution algebra, then E is a simple evolution algebra if and only if E has no algebraically transient generator.*

(3) *If E has no algebraically transient generator, then E can be written as a direct sum of Evolution subalgebras (the number of summands can be one).*

Now, for any evolution algebra, is there always an algebraically persistent generator? Generally, it is not true. For any finite dimensional evolution algebra, we have the following statement.

Theorem 3.3. *Any finite dimensional evolution algebra has a simple evolution subalgebra.*

Proof. We assume the evolution algebra E is connected, otherwise we just need to consider a component of a direct sum of E.

Let $\{g_i, g_2, \cdots, g_n\}$ be a generator set of E. Consider evolution subalgebras generated by each generator $\langle g_1\rangle$, $\langle g_2\rangle$, $\cdots\cdots$, $\langle g_n\rangle$. If there is a subalgebra that is simple, it is done. Otherwise, we choose a subalgebra which contain the least number of generators, for example, $\langle g_i\rangle$ and $\{g_{i_1}, g_{i_2}, \cdots, g_{i_k}\} \subset \langle g_i\rangle$, where $\{g_{i_1}, g_{i_2}, \cdots, g_{i_k}\}$ is a subset of $\{g_1, g_2, \cdots, g_n\}$. Then, consider $\langle g_{i_1}\rangle$, $\langle g_{i_2}\rangle$, $\cdots\cdots$, $\langle g_{i_k}\rangle$. If there is some subalgebra that is simple in this sequence, we are done. Otherwise, we choose a certain $\left\langle g_{i_j}\right\rangle$ in the same way as we choose $\langle g_i\rangle$. Since the number of generators is finite, this process will stop. Therefore, we always have a simple evolution subalgebra. $\square$

4 Hierarchy of an evolution algebra

The hierarchical structure of an evolution algebra is an interesting property that gives a picture of the dynamical process when multiplication in the evolution algebra is treated as a discrete-time dynamical step.

4.1 The periodicity of a simple evolution algebra

As we know, all generators of a simple evolution algebra have the same period. It might be well to say that a simple algebra has a period. Thus, simple evolution algebras can be roughly classified as either periodic or aperiodic. The following theorem establishes the structure of a periodic simple evolution algebra.

Theorem 4.1. *Let E be a simple evolution algebra with generator set $\{g_i \mid i \in \Lambda\}$, then all generators have the same period, denoted by d. There is a partition of generators with d disjointed classes C_0, C_2, $\cdots$, C_{d-1}, such that $\bigwedge(\Delta_k) \subseteq \Delta_{k+1}(\mod d)$, or $\Delta_k^2 \subseteq \Delta_{k+1}(\mod d)$, $k = 1, 2, \cdots d-1$,* *where* $\Delta_k = Span\,(C_k)$ *and* $\bigwedge$ *is the evolution operator,* $\mod$ *is taken with respect to the index of the class of generators. There is also a direct sum of linear subspaces*

$$E = \Delta_0 \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1}.$$

Proof. Since E is simple, if any generator g_i has a period of d, then every generator has a period of d. Set $C_m = \left\{ g_j \mid g_j \prec g_i^{[nd+m]}, j \in \Lambda \right\}$, $0 \leq m < d$, for any fixed g_i. Because this evolution algebra is simple, each generator g_j will occur in some C_m. So

$$\cup_{m=0}^{d-1} C_m = \{g_k \mid k \in \Lambda\}$$

Claim that these C_m are disjoint. We can also prove that if we take g_k as a fixed generator that is different from the previous g_i for partitioning, we still get the same partition. Now, if $g_j \in C_k$, then $g_i^{(2^{nd+k})} = ag_j + v$, $a \neq 0$. We have $g_i^{[k+1]} = a^2 g_j^2 + v^2 = a^2 \bigwedge(g_j) + v^2$, which means that generators occur in $\bigwedge(g_j) \in C_{k+1}$ or generators occur in$g_j^2 \in C_{k+1}$.

Denote the linear subspace spanned by C_k as Δ_k, $k = 1, 2, \cdots d-1$, then we have a direct sum for $E = \Delta_0 \oplus \Delta_1 \oplus \cdots \oplus \Delta_{d-1}$, and $\bigwedge: \quad \Delta_k \to \Delta_{k+1}$ for $k = 1, 2, \cdots d-1$. Or, we have $\Delta_k^2 \subseteq \Delta_{k+1}$, $\Delta_k^d \subseteq \Delta_k$ for $k = 1, 2, \cdots d-1$.

This concludes the proof. □

4.2 Semi-direct-sum decompositions of an evolution algebra

A general evolution algebra has algebraically persistent generators and algebraically transient generators. These two types of generators have distinct "reproductive behavior" — dynamical behavior. Algebraically persistent ones can generate a simple subalgebra. Once an element belongs to a subalgebra, it will never "reproduce" any element that is not in the subalgebra.

Or, dynamically, once the dynamical process, represented by the evolution operator $\bigwedge$, enters a simple evolution subalgebra, it will never escape from it. In contrast, algebraically transient generators behave differently. They generate reducible subalgebras. The following theorem demonstrates how to distinguish these two types of generators algebraically.

Theorem 4.2. *(In page 45 of [10]) Let E be a connected evolution algebra. As a vector space, E has a decomposition:*

$$E = A_1 \oplus A_2 \oplus \cdots \oplus A_n \stackrel{\bullet}{+} B$$

where A_i, $i = 1, 2, \cdots, n$, *are all simple evolution subalgebras,* $A_i \cap A_j = \{0\}$ *for* $i \neq j$, *and B is a subspace spanned by algebraically transient generators (which we call a transient space). We call this decomposition a semi-direct-sum decomposition of E. This decomposition is unique for all algebras which are evolutionary isomorphic each other.*

Note, if E is simple, n is 1 and $B = \phi$.

4.3 Hierarchy of an evolution algebra

(1) The $0-th$ structure of an evolution algebra E : the $0-th$ decomposition of E is given by the above theorem

$$E = A_1 \oplus A_2 \oplus \cdots \oplus A_{n_0} \stackrel{\bullet}{+} B_0$$

where B_0 is called the $0 - th$ transient space.

(2) The $1 - st$ structure of E is the decomposition of B_0 :

- Briefly speaking, in the space B_0, we can define every induced (or first) algebraic concepts: induced multiplication, induced evolution operator,induced evolution subalgebra, first algebraical persistency, etc.

- Theorem of decomposition of B_0 is given by

$$B_0 = A_{1,1} \oplus A_{1,2} \oplus A_{1,3} \oplus \cdots \oplus A_{1,n_1} \overset{\bullet}{+} B_1$$

where $A_{1,i}$, $i = 1, 2, \cdots, n_1$, are all first simple evolution subalgebras of B_0, $A_{1,i} \cap A_{1,j} = \{0\}$, if $i \neq j$, and B_1 is the first transient space spanned by the first algebraically transient generators.

(3) We can construct the 2nd induced evolution algebra over the first transient space B_1, if B_1 is connected and not simple. If the $k-th$ transient space B_k is disconnected and each component is simple, we will stop with a direct sum of $(k+1)-th$ simple evolution subalgebras. Otherwise, we can continue to construct evolution subalgebras until we reach a level where each evolution subalgebra is simple. Now, we have the hierarchy as follows

$$\begin{aligned}
E &= A_{0,1} \oplus A_{0,2} \oplus \cdots \oplus A_{0,n_0} \overset{\bullet}{+} B_0 \\
B_0 &= A_{1,1} \oplus A_{1,2} \oplus \cdots \oplus A_{1,n_1} \overset{\bullet}{+} B_1 \\
B_1 &= A_{2,1} \oplus A_{2,2} \oplus \cdots \oplus A_{2,n_2} \overset{\bullet}{+} B_2 \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
B_{m-1} &= A_{m,1} \oplus A_{m,2} \oplus \cdots \oplus A_{m,n_m} \overset{\bullet}{+} B_m \\
B_m &= B_{m,1} \oplus B_{m,2} \oplus \cdots \oplus B_{m,h}
\end{aligned}$$

where $A_{k,l}$ is a $k-th$ simple evolution subalgebra, $A_{k,l} \cap A_{k,l\prime} = \{0\}$ if $l \neq l\prime$, B_k is the $k-th$ transient space. B_m can be decomposed as a direct sum of $(m+1)-th$ simple evolution subalgebras. We may call these $(m+1)-th$ simple evolution subalgebras the heads of the hierarchy, h is the number of heads.

(4) This hierarchy is unique for all algebras which are evolutionary isomorphic each other.

4.4 Reducibility of an evolution algebra

From the hierarchy of an evolution algebra, we get an impression about dynamical flows of an algebra. That is, if we start at a high level, a big index level, the dynamical flow will automatically go down to a low level, it may also sojourn in a simple evolution subalgebra at each level. It is reasonable to view each simple evolution subalgebra at each level as one point or one dimensional subalgebra. The big evolutionary picture still remains. If we call this remained hierarchy the skeleton of the original evolution algebra, all evolution algebras which possess the same skeleton will have a similar dynamical characteristics, and they are evolutionary homomorphic to the skeleton. We call this procedure the reducibility of an evolution algebra and write it as a statement.

Theorem 4.3. *(In page 50 of [10]) Every evolution algebra E is evolutionary homomorphic to a unique evolution algebra E_r such that its evolution subalgebras in its hierarchy are all one dimensional subalgebras. Such a unique evolution algebra E_r is called the skeleton of E.*

The concept of the skeleton can be used to give a rough classification of all evolution algebras. The number of levels m and the numbers n_k of simple evolution subalgebras at each level k, can roughly determine the shape of the hierarchy of an evolution algebra, ignoring the flow relations between two different levels. We define skeleton-shapes to classify evolution algebras. If two evolution algebras have the same number m of levels and the same numbers n_k of simple evolution subalgebras at each level k including h subalgebras in $m+1$ level, we say these two evolution algebras belong to the same class of skeleton-shape. Furthermore, we say two evolution algebras belong to the same class of skeleton if they belong to the same class of skeleton-shape and the flow relations between any two different levels correspondingly are the same.

Given the level number m and the total number n of simple evolution subalgebras (including heads) wherever they are, how many classes of skeleton-shapes of evolution algebras can we have? The answer is a famous number in number theory, $p_{m+1}(n)$, the number of partitions of n into $m+1$ cells. For $n < m+1$, $p_{m+1}(n) = 0$ and $p_{m+1}(m+1) = 1$. Generally, we have the recursion

$$p_{m+1}(n) = p_{m+1}(n-m-1) + p_m(n-m-1) + \cdots + p_1(n-m-1).$$

Generally, we have

$$p_{m+1}(n) = \frac{n^m}{m!\,(m-1)!} + R_{m-1}(n), \;\; n \equiv n'\left((m+1)!\right),$$

where $R_{m-1}(n)$ is a polynomial in n of degree at most $m-1$. Therefore, by the number of level and the numbers of simple evolution subalgebras, we can determine an evolution algebra up to its skeleton-shape. We thus obtain a skeleton-shape classification of all evolution algebras.

In order to get a skeleton classification for all evolution algebras, we need to know how many classes of skeletons of evolution algebras we can have given the level number m and the numbers n_k of simple evolution subalgebras at each level. Set

$$bp(n,m) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} (\sum_{l=0}^{m}(-1)^l \binom{m}{l} 2^{(n-k)(m-l)}).$$

Then the number of classes of skeletons is given by

$$bp(n_0, n_1)bp(n_1, n_2) \cdots bp(n_{m-1}, n_m).$$

Therefore, by the number of levels and subalgebras at each level, we can determine an evolution algebra up to its skeleton.

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