Evolution Algebras

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Outline

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1. Introduction
2. Basic facts about evolution algebras
   - Evolution algebras
   - Product and Change of basis

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   - Subalgebras and ideals

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2. Basic facts about evolution algebras
   - Evolution algebras
   - Product and Change of basis
   - Subalgebras and ideals
   - Non-degenerate evolution algebras
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3. Decomposition of an evolution algebra
   - Ideals generated by one element
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5. Classification of three-dimensional evolution algebras

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HISTORY OF ADVANCES

1856

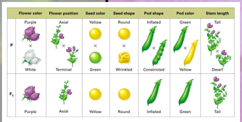
1865

1934

1936

1939

Etherington provided a precise mathematical formulation of Mendel's laws in terms of nonassociative algebras.



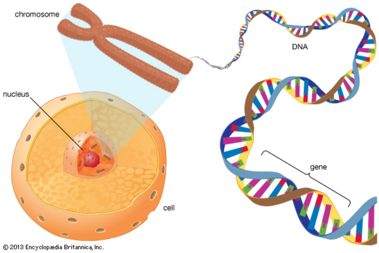
Gregor Mendel, "the father of genetics", begins detailed experiments breeding pea plants.

Mendel publishes his work describing the basic laws of inheritance.

Serebrowsky gave an algebraic interpretation of sign "x".

Glivenkov introduced the so-called Mendelian algebras.

Gene: Molecular unit of hereditary information.



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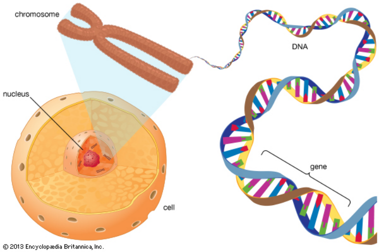
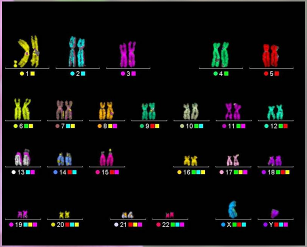
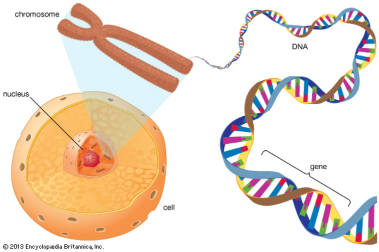


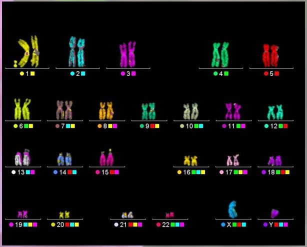


Chromosomes: Long strands of DNA formed by ordered sequences of genes. In the process of reproduction, the attributes of the offspring are inherited from alleles contained in the chromosomes of the parents.

Gene: Molecular unit of hereditary information.

Allele: Distinct forms of genes to an attribute. For example, the gene for eye color has three alleles: brown, green and blue.





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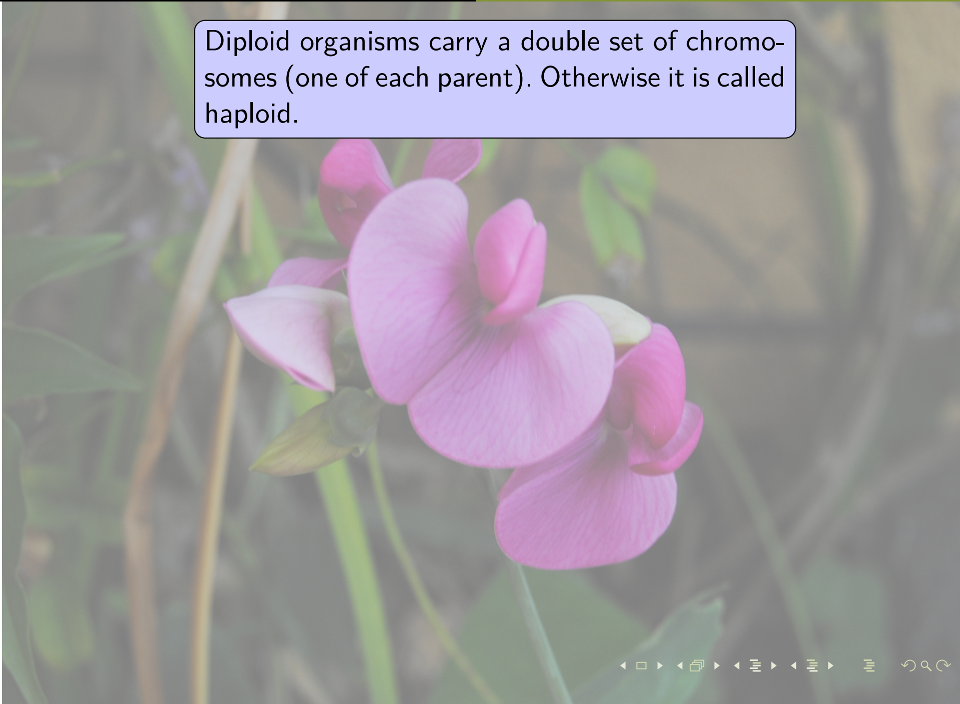

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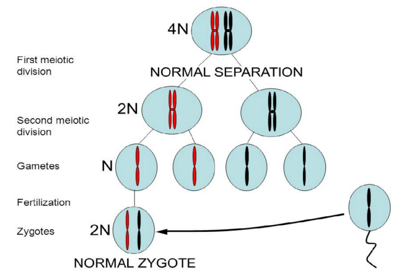

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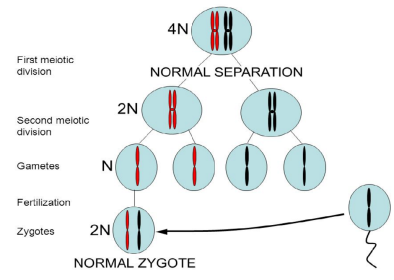

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The fusion of two gametes of opposite sex gives rise to a **zygote**, which contains a double set of chromosomes.

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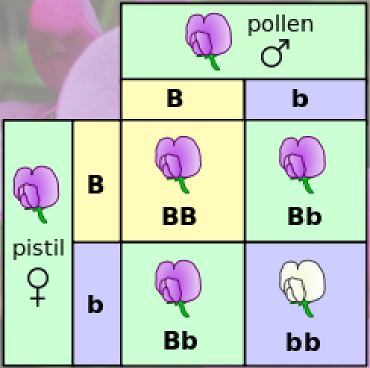

Gametic Algebra

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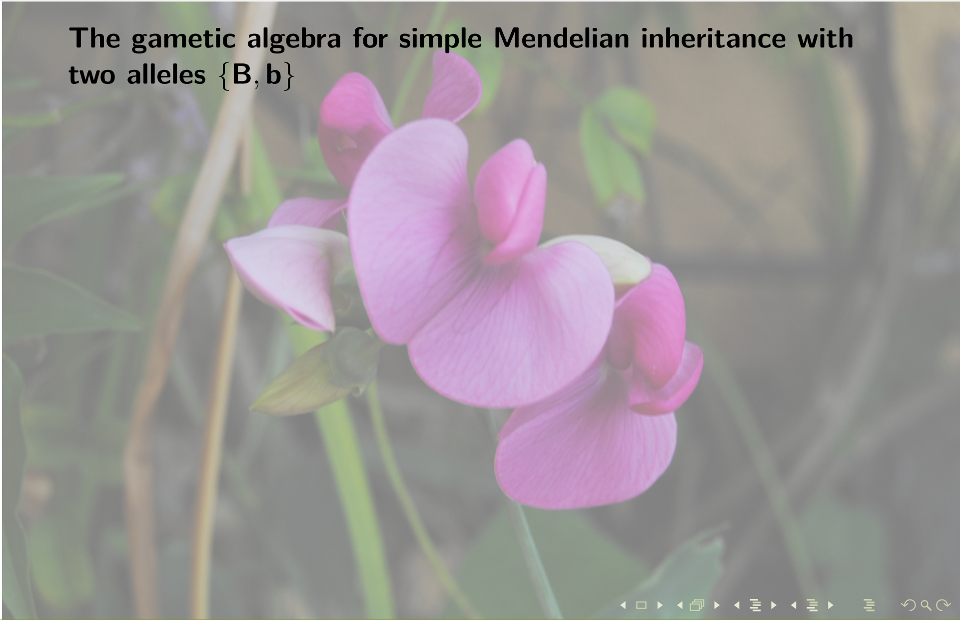
The gametic algebra for simple Mendelian inheritance with two alleles $\{B, b\}$



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Consider the set of gametes $B = \{a_1, \dots, a_n\}$ as abstract elements. Define the n dimensional algebra over $\mathbb{R}$ with basis B and multiplication

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k \text{ such that } \sum_{k=1}^{n} \gamma_{ijk} = 1.$$

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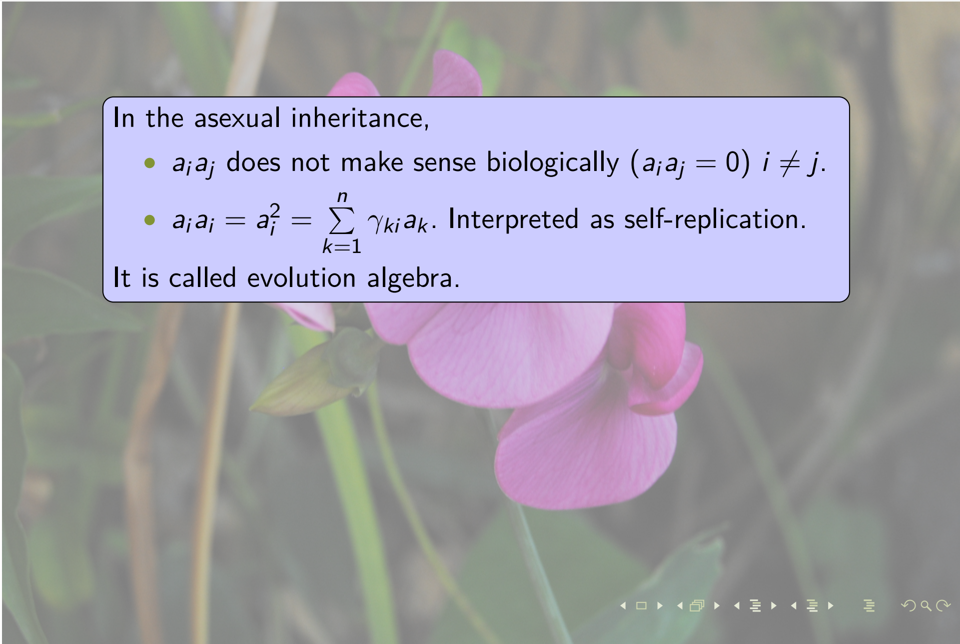

In the asexual inheritance,

- $a_i a_j$ does not make sense biologically ($a_i a_j = 0$) $i \neq j$.
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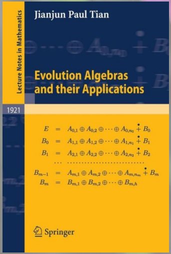

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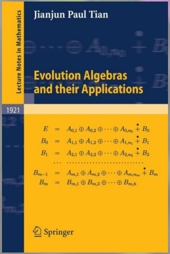

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- $\mathrm{CFM}_\Lambda(\mathbb{K}) = (M_\Lambda(\mathbb{K}), +, \cdot)$ such that for which every column has at most a finite number of non-zero entries.

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$$\xi_B(xy) = M_B\left(\xi_B(x) \bullet_B \xi_B(y)\right),$$

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Let A be an evolution algebra and let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of A with structure matrix $M_B = (\omega_{ij})$.

1. If $B' = \{f_i \mid i \in \Lambda\}$ is a natural basis of A with $P_{B'B} = (p_{ij})$ the change of basis matrices, then

Change of basis

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Let A be an evolution algebra and let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of A with structure matrix $M_B = (\omega_{ij})$.

1. If $B' = \{f_i \mid i \in \Lambda\}$ is a natural basis of A with $P_{B'B} = (p_{ij})$ the change of basis matrices, then

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Evolution subalgebras. Evolution ideals

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3. Not every ideal of an evolution algebra has a natural basis.

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Remark

Does non-degeneracy depend on the considered natural basis?

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4. $|\Lambda_0(B)| = |\Lambda_0(B')|$ for every natural basis B' of A.

Consequently, the definition of non-degenerate evolution algebra does not depend on the considered natural basis.

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Let I be an ideal of an evolution algebra A. I has the **absorption property** if $xA \subseteq I$ implies $x \in I$.

Lemma

- An ideal I of an evolution algebra A has the absorption property if and only if $\mathrm{ann}(A/I) = \overline{0}$.
- If I is a non-zero ideal which it has the absorption property, then I is an evolution ideal and has the extension property.

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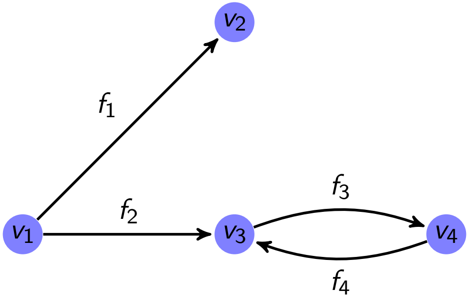
Corollary

Let I be an ideal of an evolution algebra A. Then $A/\mathrm{rad}(A)$ is a non-degenerate evolution algebra.

Basic concepts about graphs

Consider the following graph E:



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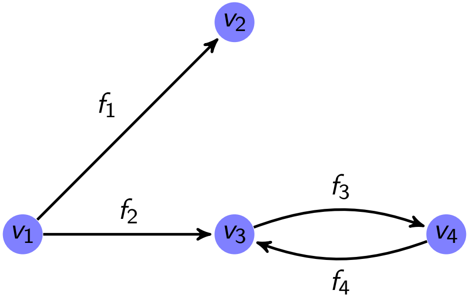

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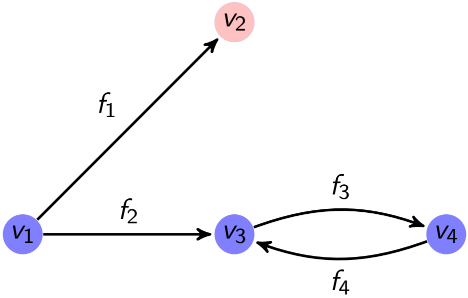

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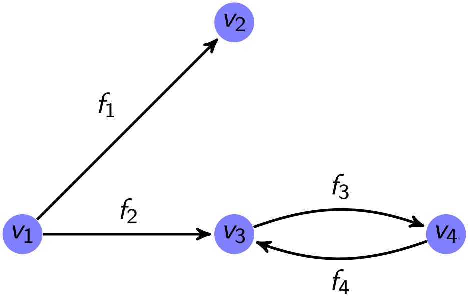

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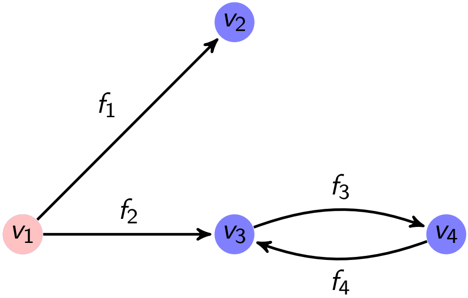

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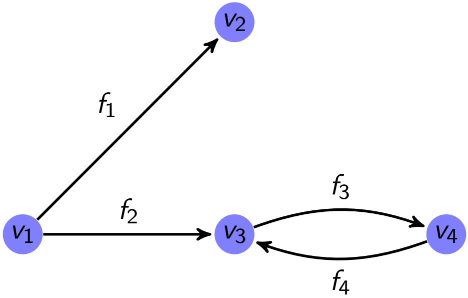

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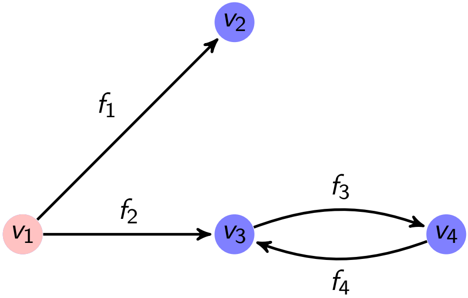

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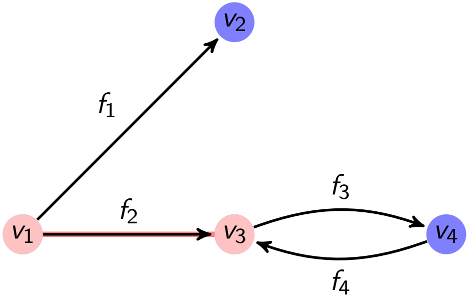

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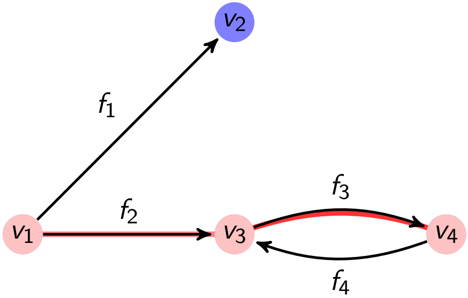

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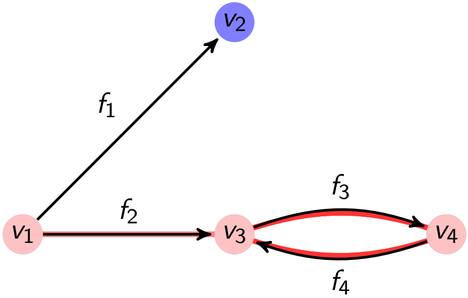

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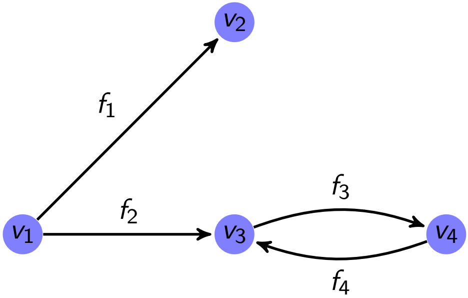

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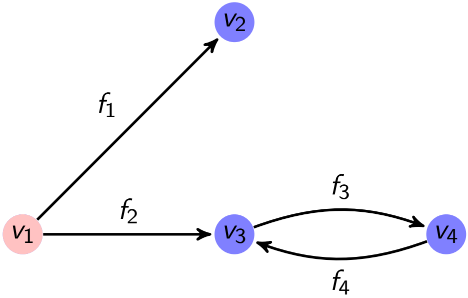

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- Path $(E) = \{f_1, f_2, f_3, f_4, f_2 f_3, f_3 f_4, f_2 f_3 f_4\}$.

- Adjacency matrix

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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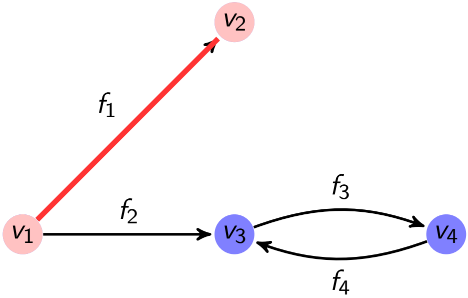

- **Condition Sing**.
- Sink $\Rightarrow$ v_2.
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- Path $(E) = \{f_1, f_2, f_3, f_4, f_2 f_3, f_3 f_4, f_2 f_3 f_4\}$.

- Adjacency matrix

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & {\color{red}1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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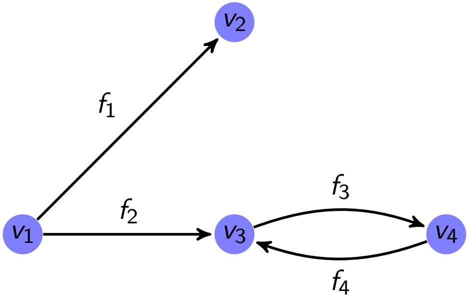

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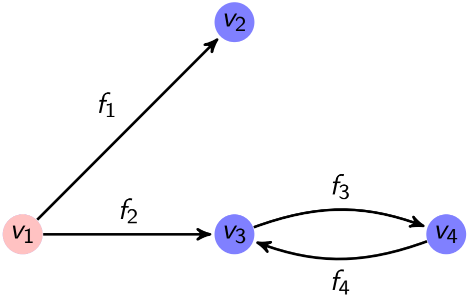

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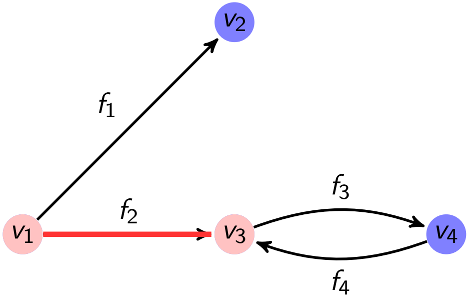

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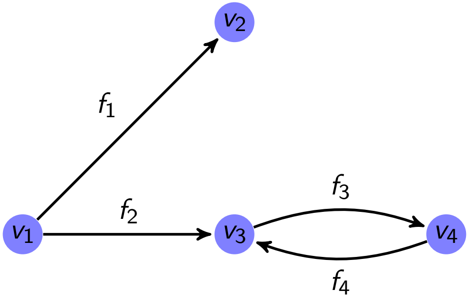

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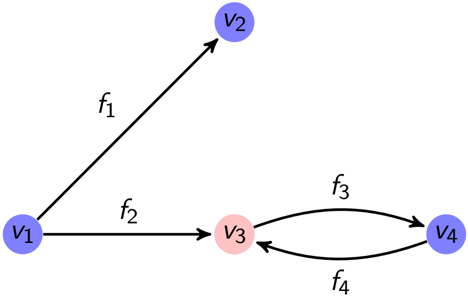

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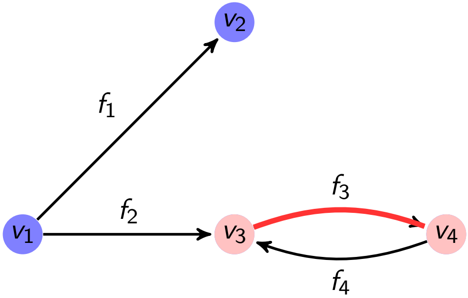

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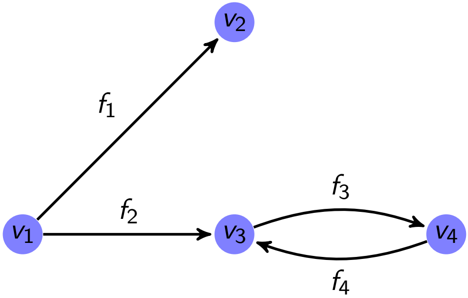

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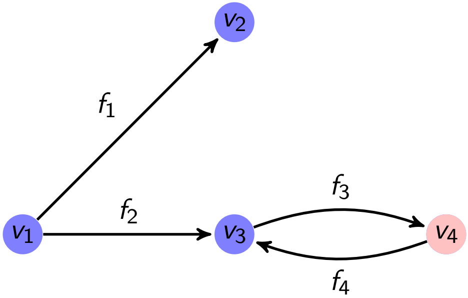

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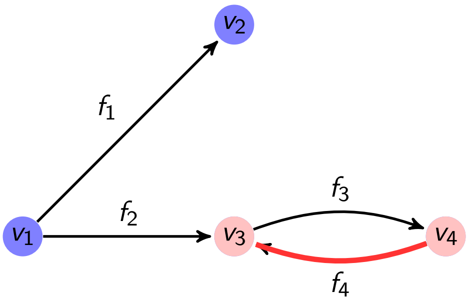

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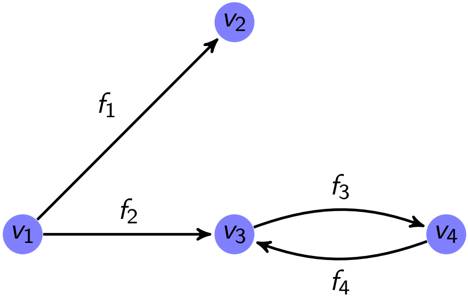

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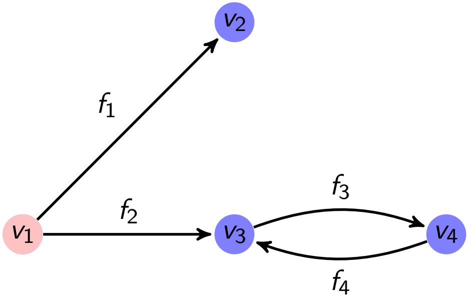

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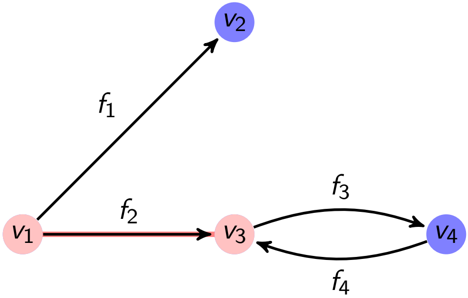

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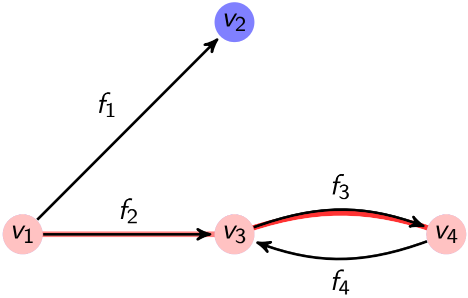

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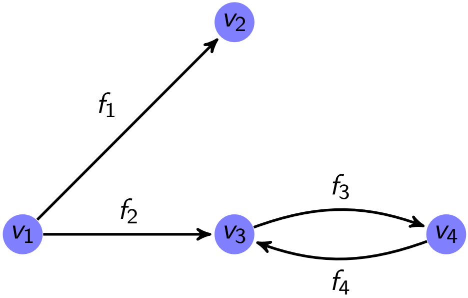

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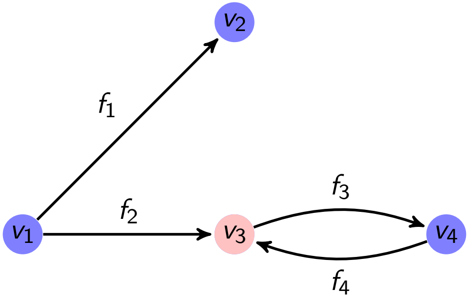

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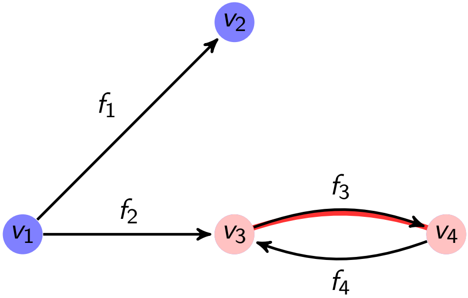

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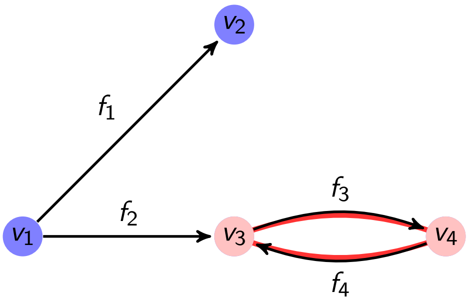

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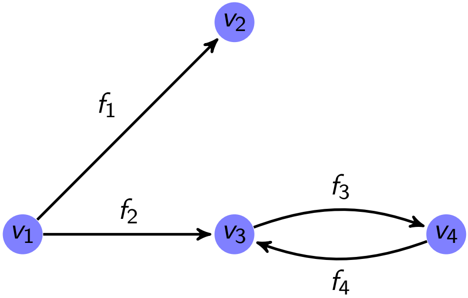

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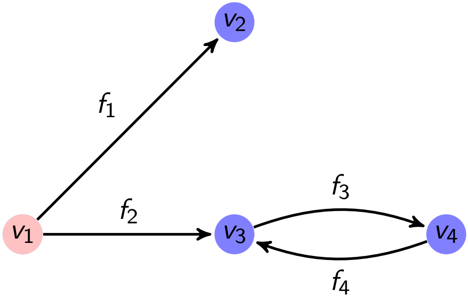

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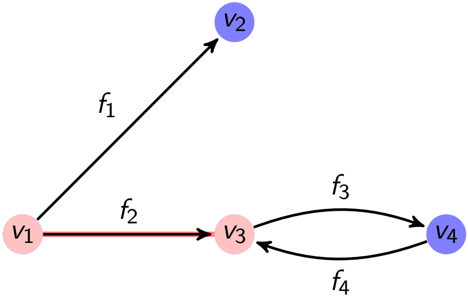

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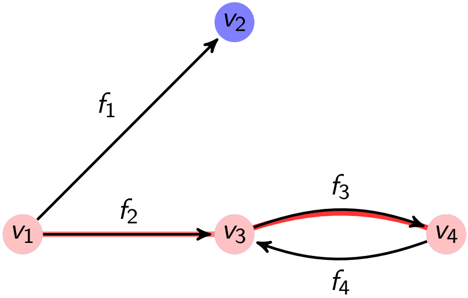

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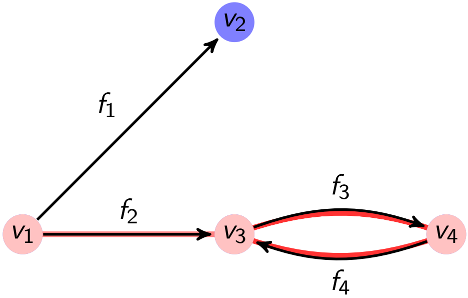

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Graph associated

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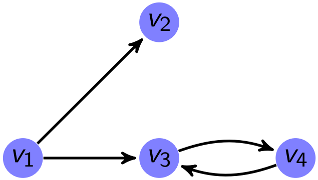

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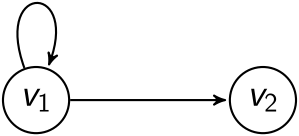
Let A be the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1 + e_2$ and $e_2^2 = 0$. Consider the natural basis $B' = \{e_1 + e_2, e_2\}$. Then the graphs associated to the bases B and B' are, respectively:

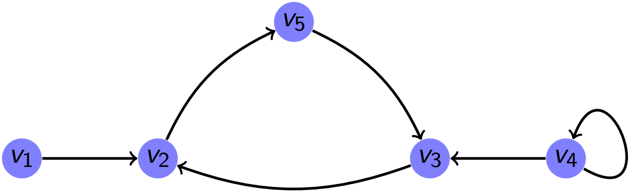
E:



F:

Outline

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$



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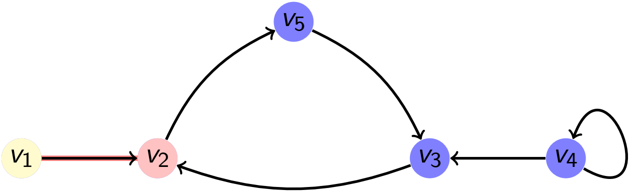

- $D^1(1) = \{k \in \Lambda \mid e_1^2 = \sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2\}$.

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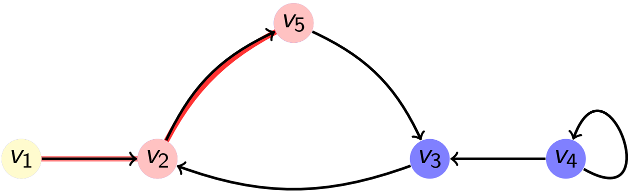
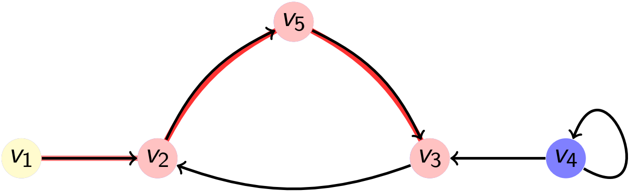


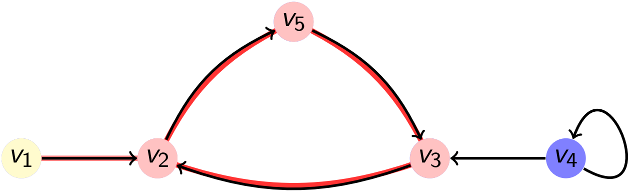
- $D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{5\}.$

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$



- $D^3(1) = \{3\}$.

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$



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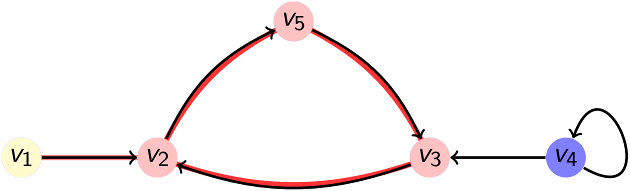

- $D(1) = \bigcup_{m\in\mathbb{N}} D^m(1) = \{2, 5, 3\}$.

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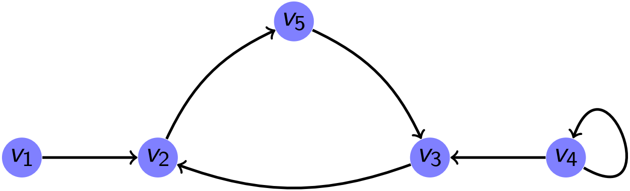

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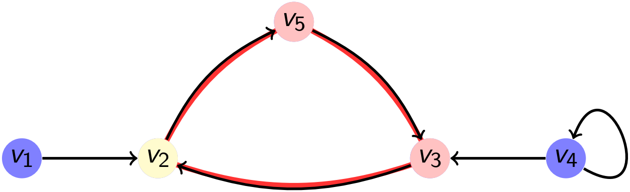

- $C(2) = \{j \in \Lambda \mid j \in D(2) \text{ and } 2 \in D(j)\} = \{2, 5, 3\}$.

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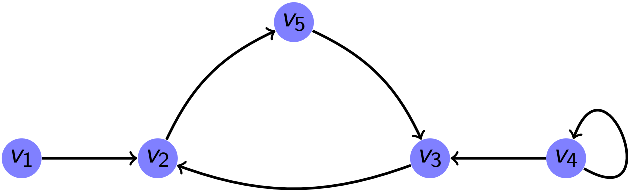

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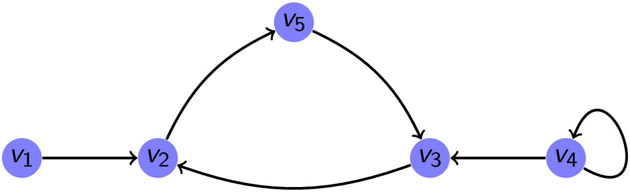

- A cyclic index i_0 is a **principal cyclic index** if $j \in D(i_0)$ for every $j \in \Lambda$ with $i_0 \in D(j)$.

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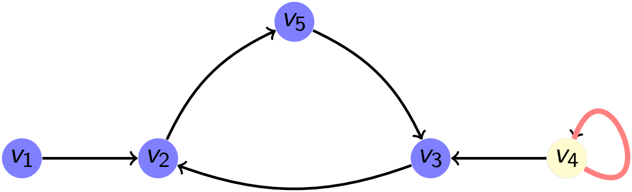

- 4 is a principal cyclic index.

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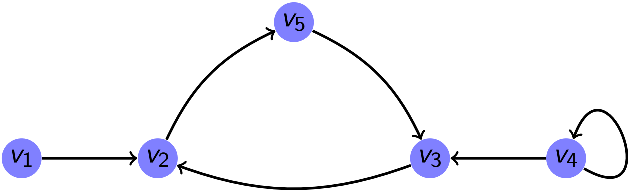

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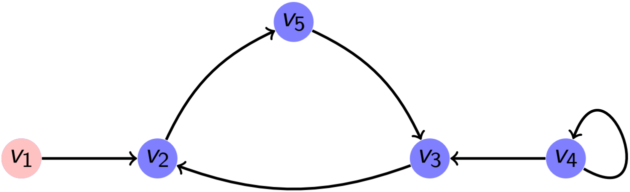

- 1 is a chain-start index because 1 has no ascendents.

Ideal generated by one element

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Corollary

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Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis.

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Let A be an evolution algebra. For any element $x \in A$,

$\dim \langle x \rangle$ is at most countable.

Simple evolution algebras

Simple evolution algebras

Definition

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An algebra A is **simple**

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Moreover, the dimension of A is at most countable and if $|\Lambda| < \infty$ then the converse is true.

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 & \vdots & & & & \vdots &
\end{array}
$$

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 & \vdots & & & & \vdots &
\end{array}$$

Then A satisfies the conditions 1, 2 and 3 but A is not simple as $\langle e_1^2 \rangle$ and $\langle e_2^2 \rangle$ are two non-zero proper ideals.

Characterization for finite dimension

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Let A an evolution algebra with dim(A)=n and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A.

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$$\begin{pmatrix} W_{m\times m} & U_{m\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}$$

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for some $m \in \mathbb{N}$ with $m < n$ and matrices $W_{m\times m}$, $U_{m\times(n-m)}$ *and* $Y_{(n-m)\times(n-m)}$.

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Let A be a non-degenerate evolution algebra with a natural basis $B = \{e_i \mid i \in \Lambda\}$. Assume that $A = \oplus_{\gamma \in \Gamma} I_\gamma$, where each I_γ is an ideal of A. Then, there exists a disjoint decomposition of Λ, say $\Lambda = \sqcup_{\gamma \in \Gamma} \Lambda_\gamma$ such that

$$I_\gamma = \mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}.$$

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$$M_{B'} = \begin{pmatrix} W_{m\times m} & 0_{(n-m)\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}$$

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Corollary

Let A be a non-degenerate evolution algebra, $B = \{e_i \mid i \in \Lambda\}$ a natural basis and let E be its associated graph.

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Corollary

Let A be a non-degenerate evolution algebra, $B = \{e_i \mid i \in \Lambda\}$ a natural basis and let E be its associated graph. Then A es irreducible if and only if E is a connected graph.

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Theorem

Let A be a non-degenerate evolution algebra. Then A admits an optimal direct-sum decomposition. Moreover, it is unique.

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2. $\Lambda_i \cap \Lambda_j = \emptyset$, for every $i, j \in \{1, \ldots, k\}$, with $i \neq j$.

If for every $i \in \{1, \ldots, k\}$ the index set $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ is not fragmentable then we say that $\Lambda = \cup_{i=1}^{k} \Lambda_i$ is an optimal fragmentation.

The optimal direct-sum decomposition of A

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where $\Lambda(S) := S \cup_{i \in S} D(i)$.

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Let $\Lambda = \sqcup_{\gamma \in \Gamma} \Lambda_\gamma$ be the optimal fragmentation of $(\dagger)$ and decompose $B = \sqcup_{\gamma \in \Gamma} B_\gamma$, where $B_\gamma = \{e_i \mid i \in \Lambda_\gamma\}$. Then $A = \oplus_{\gamma \in \Gamma} I_\gamma$, for $I_\gamma = \mathrm{lin}\ B_\gamma$, which is an evolution ideal of A. Moreover, if A is non-degenerate, then $A = \oplus_{\gamma \in \Gamma} I_\gamma$ is the optimal direct-sum decomposition of A.

Optimal Fragmentation

PROGRAM 1

Program A

In[1]:=

$$P = \begin{pmatrix} 0&0&0&0&1&0&0&1\\ 0&0&1&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&1&1 \end{pmatrix};$$

```
D1[i_, P_] := Select[Table[j, {j, Length[P]}], P[[#, i]] ≠ 0 &];
D1[5, P]
Dm_[i_, P_] := Module[{a, k, s}, a = {}; s = Length[Dm-1[i, P]];
    If[m == 1, D1[i, P],
     Union[Flatten[
       Table[D1[Dm-1[i, P][[t]], P], {t, Length[Dm-1[i, P]]}]]]]]];

CycleQ[P_] := Module[{n, x}, n = Length[P];
    x = Union[Flatten[Table[
         Diagonal[MatrixPower[P, i]], {i, 1, n}]]];
    MemberQ[x, 1]];
CycleQ[P]

DYesCycle[i_, P_] := Module[{b, j},
    b = {};
    For[j = 1, j ≤ Length[P], j++, AppendTo[b, Dj[i, P]]];
    Apply[Union, b]];

DNotCycle[i_, P_] := Module[{t, b},
    b = {D1[i, P]};
    For[t = 1, Dt[i, P] ≠ Dt+1[i, P], t++,
     AppendTo[b, Dt+1[i, P]]
    ];
    Apply[Union, b]];

DP[i_, P_] := If[CycleQ[P], DYesCycle[i, P], DNotCycle[i, P]]
```

Out[3]= {1, 5}

Out[6]= True

In[10]:=

```
DP[5, P]
```

Out[10]= {1, 2, 5, 7, 8}

Program B

In[11]:=

```
CyclicQ[i_, P_] := If[MemberQ[DP[i], i],
   Print[i "is a cyclic index"], Print[i " is not a cyclic index"]];
CycleAssociated[i_, P_] := Select[Table[j, {j, Length[P]}],
   MemberQ[DP[i, P], #] && MemberQ [DP[#, P], i] &];
Ascendents[i_, P_] := Module[{j, b}, b = {};
   For[j = 1, j ≤ Length[P], j ++, If[MemberQ[DP[j, P], i], AppendTo[b, j]]]; b];
Subset[A_, B_] := (Union[A, B] == Union[B]);
PrincipalCycleQ[i_, P_] :=
  If[Subset[Ascendents[i, P], CycleAssociated[i, P]],
   Print[i "is a principal cyclic-index"],
   Print[i "is not a principal cyclic-index"]];
ElementsNotNoneRow[P_] := Module[{j},
   Select[Table[j, {j, Length[P]}], P[[#]] == 0 P[[1]] &]];
(* La función incad me devuelve los índices i tales que la fila i es nula*)
ChainStartQ[i_, P_] :=
  If[MemberQ[ElementsNotNoneRow[P], i], Print[i "is a chain-start index"],
   Print[i " is not a chain-start index"]];
CycleAssociated[5, P]
Ascendents[5, P]
```

Out[18]= {5}

Out[19]= {5}

Program C

In[20]:=

```
LambdaPrincipalCycle[P_] := Module[{j, b},
   b = {};
   For[j = 1, j ≤ Length[P], j ++,
    If[
     Subset[Ascendents[j, P], CycleAssociated[j, P]], AppendTo[b, DP[j, P]]]]
   ; b];
Λ[i_, P_] := Union[{i}, DP[i, P]];
LambdaChainStart[P_] :=
 Table[Λ[ElementsNotNoneRow[P][[i]], P], {i, Length[ElementsNotNoneRow[P]]}]
CanonicalDecomposition[P_] :=
  Join[LambdaChainStart[P], LambdaPrincipalCycle[P]];
CanonicalDecomposition[P]
```

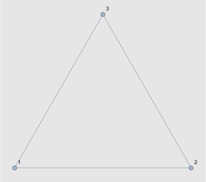
Out[24]= {{2, 3, 4, 6}, {1, 2, 5, 7, 8}, {2, 3, 4}}

PROGRAM 2

programa dimension 8.nb 3

In[25]:=
```
f[i_, j_, P_] := If[i == j, 0, If[Intersection[Part[CanonicalDecomposition[P], i],
      Part[CanonicalDecomposition[P], j]] ≠ {}, 1, 0]];
Matr[P_] := Table[f[i, j, P], {i, Length[CanonicalDecomposition[P]]},
   {j, Length[CanonicalDecomposition[P]]}];
AdjacencyGraph[Matr[P], VertexLabels → "Name"]
OptimalFragmentation[P_] :=
 ConnectedComponents[AdjacencyGraph[Matr[P], VertexLabels → "Name"]]
OptimalFragmentation[P]
```

Out[27]=



Out[29]= (1 2 3)

Outline

Classification of 2-dimensional complex evolution algebras

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Theorem (Casas J.M., Ladra M., Omirov B.A., and Rozikov U.A., 2014)

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Any 2-dimensional complex evolution algebra E is isomorphic to one of the following pairwise non isomorphic algebras:

1. dim E^2=1
   - E_1: $e_1 e_1 = e_1$,
   - E_2: $e_1 e_1 = e_1$, $e_2 e_2 = e_1$,
   - E_3: $e_1 e_1 = e_1 + e_2$, $e_2 e_2 = -e_1 - e_2$,
   - E_4: $e_1 e_1 = e_2$.

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   - E_4: $e_1e_1 = e_2$.
2. dim E^2=2
   - E_5: $e_1e_1 = e_1 + a_2e_2$, $e_2e_2 = a_3e_1 + e_2$, $1 - a_2a_3 \neq 0$, where $E_5(a_2, a_3) \cong E_5'(a_3, a_2)$,
   - E_6: $e_1e_1 = e_2$, $e_2e_2 = e_1 + a_4e_2$, $a_4 \neq 0$, where $E_6(a_4) \cong E_6(a_4')$ $\Leftrightarrow \frac{a_4'}{a_4} = \cos\frac{2\pi k}{3} + \imath \sin\frac{2\pi k}{3}$ for some $k = 0, 1, 2$.

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If $\dim(A^2) = 1$ then M_B is one of the following four matrices:

1. $M_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,
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3. $M_B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,
4. $M_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   They are mutually non-isomorphic.

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5. $M_B(\alpha, \beta) = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{K}^\times$ and $1 - \alpha\beta \neq 0$.
6. $M_B(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ for some $\alpha \in \mathbb{K}^\times$.
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They are mutually non-isomorphic except in the case $\{M_B(\gamma) \mid \gamma \in \mathbb{K}\}$ when $\frac{\gamma}{\gamma'}$ is a $3rd$ root of unity.

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Fix a two-dimensional evolution algebra A and a natural basis $B = \{e_1, e_2\}$. Let

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- Suppose that $\{e_1^2\}$ is a basis of A^2. Since $e_2^2 \in A^2$, there exists $c_1 \in \mathbb{K}$ such that $e_2^2 = c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2)$.

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- Extend the basis $\{e_1^2\}$ of A^2 to a new basis $B' = \{e_1', e_2'\}$ of A (B' is not necessarily a natural basis) with change of basis matrix $P_{B'B}$

$$P_{B'B} = \begin{pmatrix} \omega_1 & p_1 \\ \omega_2 & p_2 \end{pmatrix}.$$

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- Distinguish different cases depending on $\omega_1 p_1 + \omega_2 p_2 c_2$ is zero or not.

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- We study if the parametric families of evolution algebras are isomorphic when we change of parameters.

Outline

Action of $S_3 \ltimes (\mathbb{K}^\times)^3$ on $\mathrm{M}_3(\mathbb{K})$

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6. The action of P on M can be formulated as follows:

$$P \cdot M := P^{-1} M P^{(2)}.$$

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3. The rank of M and the rank of $P \cdot M$ coincide.
4. Let M_B be the structure matrix of an evolution algebra A such that $A^2 = A$. If N is the structure matrix of A relative to a natural basis B' then there exists $Q \in S_3 \rtimes (\mathbb{K}^\times)^3$ such that $N = Q \cdot M_B$.

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✓ If $\dim(A^2) = 0$ then $M_B = 0$ for any natural basis B of A.

Three-dimensional evolution algebras

Let A be a three-dimensional evolution $\mathbb{K}$-algebra where $\mathbb{K}$ is a field of characteristic different from 2 and such that for any $k \in \mathbb{K}$ the polynomial of the form $x^n - k$ has a root whenever $n = 2, 3, 7$.

✓ If $\dim(A^2) = 0$ then $M_B = 0$ for any natural basis B of A.

✓ If $\dim(A^2) = 1$. Let $M_B = (\omega_{ij})$ be the structure matrix.

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✓ If $\dim(A^2) = 0$ then $M_B = 0$ for any natural basis B of A.

✓ If $\dim(A^2) = 1$. Let $M_B = (\omega_{ij})$ be the structure matrix.

- We may assume $e_1^2 \neq 0$.

$$\begin{aligned} e_1^2 &= \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \\ e_2^2 &= c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3) \\ e_3^2 &= c_2 e_1^2 = c_2(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3). \end{aligned}$$

- We analyze when A^2 has the extension property, i.e., if there exists a natural basis $B' = \{e_1^2, e_2', e_3'\}$ of A with

$$P_{B'B} = \begin{pmatrix} \omega_1 & \alpha & \delta \\ \omega_2 & \beta & \nu \\ \omega_3 & \gamma & \eta \end{pmatrix}$$

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- The conditions are as follows:

$$\begin{aligned} \alpha\omega_1 + \beta\omega_2 c_1 + \gamma\omega_3 c_2 &= 0 \\ \delta\omega_1 + \nu\omega_2 c_1 + \eta\omega_3 c_2 &= 0 \\ \alpha\delta + \beta\nu c_1 + \gamma\eta c_2 &= 0 \\ |P_{B'B}| &\neq 0 \end{aligned}$$

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- A^2 has the extension property if and only if

$$\omega_1^2 + \omega_2^2 c_1 + \omega_3^2 c_2 \neq 0$$

.

$\dim(A^2) = 1$

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Type			
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

$\dim(A^2) = 1$

Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			
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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
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$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I = < e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I = < e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	$I =< e_3 >$

$\dim(A^2) = 2$

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$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix}$$

$\dim(A^2) = 2$

✓ If $\dim(A^2) = 2$. We may assume that there exists a natural basis $B = \{e_1, e_2, e_3\}$ such that

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix}$$

for some $c_1, c_2 \in \mathbb{K}$ with $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$.

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$$\begin{cases} \omega_{11}p_{11}p_{12} + \omega_{12}p_{21}p_{22} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} = & 0 \\ \omega_{21}p_{11}p_{12} + \omega_{22}p_{21}p_{22} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} = & 0 \\ \omega_{31}p_{11}p_{12} + \omega_{32}p_{21}p_{22} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{32} = & 0 \end{cases}$$

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- We distinguish several cases depending on c_1 and c_2.

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- We list the results in Tables where we apply the action of σ on the structure matrix.
- We study what happen when we change the parameters. Are the corresponding evolution algebras isomorphic?

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	

Four non-zero entries

- There are 3(15-6-3)=18 cases.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&0&0\\0&1&1\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}c_1&1&0\\0&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&c_1&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&c_2\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&c_2\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_2&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\0&c_2&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_2&0&1\end{pmatrix}$	$\begin{pmatrix}1&c_2&0\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&c_1\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\1&c_1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_1&1&0\end{pmatrix}$	$\begin{pmatrix}0&c_1&1\\1&0&1\\0&0&0\end{pmatrix}$

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	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&0&0\\0&1&1\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}c_1&1&0\\0&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&c_1&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&c_2\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&c_2\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_2&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\0&c_2&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_2&0&1\end{pmatrix}$	$\begin{pmatrix}1&c_2&0\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&c_1\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\1&c_1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_1&1&0\end{pmatrix}$	$\begin{pmatrix}0&c_1&1\\1&0&1\\0&0&0\end{pmatrix}$
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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$				

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	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			

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	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&0&0\\0&1&1\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}c_1&1&0\\0&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&c_1&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&c_2\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&c_2\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_2&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\0&c_2&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_2&0&1\end{pmatrix}$	$\begin{pmatrix}1&c_2&0\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&c_1\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\1&c_1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_1&1&0\end{pmatrix}$	$\begin{pmatrix}0&c_1&1\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&1&0\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\1&0&1\end{pmatrix}$		

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	

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	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Mutually isomorphic evolution algebras

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M_B	$P_{B'B}$	$M_{B'}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	

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M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$
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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	

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$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & \frac{1}{c_1} \end{pmatrix}$

$c_1 = 0$ and $c_2 \neq 0$

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- We have

$$\begin{aligned} p_{11}p_{12} &= 0; \quad & p_{21}p_{22} &= -c_2 p_{31}p_{32}; \\ p_{11}p_{13} &= 0; \quad & p_{21}p_{23} &= -c_2 p_{31}p_{33}; \\ p_{12}p_{13} &= 0; \quad & p_{22}p_{23} &= -c_2 p_{32}p_{33}. \end{aligned}$$

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- The possible change of basis matrices are:

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$$\left\{ \begin{pmatrix} p_{11} & 0 & 0\\ 0 & p_{22} & 0\\ 0 & 0 & p_{33} \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0\\ 0 & 0 & p_{23}\\ 0 & p_{32} & 0 \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0\\ 0 & p_{22} & \frac{-p_{32}p_{33}}{p_{22}}\\ 0 & p_{32} & p_{33} \end{pmatrix} \right\}.$$

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$c_1 = 0$ and $c_2 \neq 0$

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- We focus on the first two families of matrices (they leave invariant the number of non-zero entries in the first and second columns).
- Are the resulting families of evolution algebras included into other families?

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- We see if their evolution algebras are mutually isomorphic.

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- Are the parametric families evolution algebras isomorphic?.

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Outline

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- The different methods use to obtain the classification can be generalized to arbitrary finite-dimensional evolution algebras.

"The lack of real contact between mathematics and biology is either a tragedy, a scandal or a challenge, it is hard to decide which."

Gian Carlo Rota

Thanks!