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Dynamics of cholera epidemic models in fluctuating environments

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Based on our deterministic models for cholera epidemics, we propose a stochastic model for cholera epidemics to incorporate environmental fluctuations which is a nonlinear system of Itô stochastic differential equations. We conduct an asymptotical analysis of dynamical behaviors for the model. The basic stochastic reproduction value $\mathcal{R}_s$ is defined in terms of the basic reproduction number R_0 for the corresponding deterministic model and noise intensities. The basic stochastic reproduction value determines the dynamical patterns of the stochastic model. When $\mathcal{R}_s < 1$, the cholera infection will extinct within finite periods of time almost surely. When $\mathcal{R}_s > 1$, the cholera infection will persist most of time, and there exists a unique stationary ergodic distribution to which all solutions of the stochastic model will approach almost surely as noise intensities are bounded. When the basic reproduction number R_0 for the corresponding deterministic model is greater than 1, and the noise intensities are large enough such that $\mathcal{R}_s < 1$, the cholera infection is suppressed by environmental noises. We carry out numerical simulations to illustrate

our analysis, and to compare with the corresponding deterministic model. Biological implications are pointed out.

Keywords: Cholera; Itô stochastic differential equations; basic stochastic reproduction value; stationary ergodic distribution.

AMS Subject Classification: 92D30, 37H15, 60H10, 60J99

1. Introduction

Cholera is an infection of small intestine caused by the gram-negative bacterium called Vibrio cholerae, characterized by a variety of diarrhea, abdominal cramp and dehydration. It has caused seven global pandemics since 1816 [6, 22, 24]. The current pandemic started from Indonesia in 1961, spread into Europe, South Pacific and Japan in the late 1970s, reached South America in 1990s, arrived to Africa 2000s, and has continued to the present. The most recent case was the cholera outbreak in Yemen 2018, which caused 1,207,596 infections and 2,510 associated deaths [24]. Cholera has posed a major practical problem in the need to provide models of the epidemic reliably accounting for real-life epidemiological and environmental complexity in a dependable predictive fashion. In fact, there have been many efforts in modeling cholera epidemics. The earliest mathematical model was proposed by Capasso and Paveri-Fontana [5] to study the 1973 cholera epidemic in the Mediterranean region. Codeco extended this model to count for the role of Vibrio cholerae in the aquatic resource [7]. Based on these models, many deterministic models have been proposed, Hartly *et al.* [10], Mukandavire *et al.* [18], Jenson *et al.* [11], Pascual *et al.* [20], Nishiura *et al.* [19], to name a few. A good summary was presented in our work [23] which emphasized the global stability of these deterministic cholera epidemic models.

It is known that there are many stochastic elements or uncertainties in modeling of cholera epidemics. Actually, May pointed out in ecosystem modeling that the environmental noise affects population growth rates, carrying capacities, and other parameters [17]. Environments are deeply involved in cholera epidemics. The functional relations among variables such as susceptibles, infected individuals, and concentrations of the pathogen in water resource are subject to environmental fluctuations. Of course, the parameters in the models are not only subject to measurement errors but also to intrinsically environmental noises. Although it is insightful to obtain average dynamics of cholera epidemics from deterministic models, it is needed to have more realistic mathematical models which may provide more complete pictures for cholera epidemics. There are very few stochastic models for cholera epidemics in terms of stochastic differential equations. Gani *et al.* used a particular Poisson process to model cholera bacteria and infective numbers [8]. Bertuzzo *et al.* provided an individual-based spatially-explicit stochastic process to model cholera spreading [4]. Azaele *et al.* randomized infected individuals to obtain an analytically tractable stochastic model for cholera epidemics [2]. In this study, we

will randomize the incidence function and some parameters in a general model we proposed in [23].

In our work [23], we studied the following deterministic cholera epidemic model with a general nonlinear environment-dependent incidence function $f(B)$:

$$\begin{aligned}
\frac{dS}{dt} &= bN - S(t)f(B(t)) - bS(t),\\
\frac{dI}{dt} &= S(t)f(B(t)) - (\gamma + b)I(t),\\
\frac{dR}{dt} &= \gamma I(t) - bR(t),\\
\frac{dB}{dt} &= eI(t) - mB(t),
\end{aligned} \tag{1.1}$$

where $S(t)$ stands for the susceptible individuals at time t, $I(t)$ represents the infected individuals, $R(t)$ represents the recovered individuals and $B(t)$ stands for the concentration of the vibrios (V. cholerae) in the water resource. The parameter N is the total human population under consideration; b is the natural human birth or death rate; γ is the rate of recovery from cholera; e is the rate of contribution (e.g., V. cholerae shredding) to the aquatic resource; m is the net death rate of vibrios in the environment.

To incorporate environmental fluctuations and noises in parameter values (called them errors simply), we may consider as follows. Let P be a parameter or an incidence function (on the average), and $P + E$ be its value with errors. When the system changes from time t to $t+dt$ in a small time subsequent interval dt, the error will be Edt. From the central limit theorem, Edt can be approximated by a normal distribution with mean zero and variance $\tau^2 dt$. This way, Edt can be approximated by $\tau dW(t)$, where $dW(t) = W(t + dt) - W(t)$ and $W(t)$ is the Brownian motion following the standard normal distribution, and τ is called the intensity of the noise. Loosely speaking, we may use white noise $\frac{dW(t)}{dt}$ to model simultaneous errors.

In the deterministic model (1.1), the incidence function $f(B)$ encodes the interaction between human and cholera vibrios in the environment such as water resources. The environmental fluctuation will influence the incidence function in several ways. For example, the temperature will impact human drinking behavior and patterns of washing or swimming. The environmental factors also impact the death rate m of cholera vibrios in the water resource. From a long term viewpoint, the human birth or death rate b in a region is also influenced by its environmental situation. Particularly, when a cholera epidemic occurs in the region, the human birth/death rate can be affected. In order to count for the environmental fluctuations and random noises in parameter values, we consider the situation in which the incidence function $f(B)$, the parameter b and m are subject to white noise perturbations. We may consider each infected individual contributes environmental

noise with intensity of τ_1. Then, the incidence function $F(B)$ which is actually a rate of interactions can be replaced by $f(B)+\tau_1\frac{I}{N}\frac{dW_1}{dt}$. For the human birth/death rate b, the environmental noise is mostly from infected and susceptible individuals. Similarly, if we consider each individual of infected and susceptible people contributes environmental noise with intensity τ_2, we may replace b by $b+\tau_2\frac{SI}{N^2}\frac{dW_2}{dt}$. For the death rate of cholera vibrios, we consider the environment affects cholera vibrios as a whole in the water. So, we may replace m by $m+\tau_3\frac{dW_3}{dt}$, where τ_3 is the noise intensity in the water. It should be noticed that $\frac{dW_1}{dt}$, $\frac{dW_2}{dt}$, and $\frac{dW_3}{dt}$ are independent Gaussian white noises.

Then, we have a stochastic model in terms of Itô stochastic differential equations to describe the epidemics of cholera as follows:

$$\begin{aligned} dS &= [bN - Sf(B) - bS]dt - \tau_1\frac{SI}{N}dW_1 + \tau_2\frac{SI}{N^2}(N-S)dW_2,\\ dI &= [Sf(B) - (\gamma+b)I]dt + \tau_1\frac{SI}{N}dW_1 - \tau_2\frac{SI^2}{N^2}dW_2,\\ dR &= (\gamma I - bR)dt - \tau_2\frac{SIR}{N^2}dW_2,\\ dB &= (eI - mB)dt - \tau_3 B dW_3, \end{aligned} \tag{1.2}$$

where W_1, W_2, and W_3 are mutually independent Wiener processes and τ_1, τ_2, and τ_3 are their intensities. Here, the total population N becomes a random variable and $N = S(t) + I(t) + R(t)$. Adding the first three equations of the system (1.2) gives

$$dN = (bN - b(S+I+R))dt + \tau_2\frac{SI}{N^2}(N-S-I-R)dW_2.$$

This means that $\frac{dN}{dt} = 0$ with probability 1 and hence N is an almost sure constant random variable. For convenience of discussion, we will drop the equation for the recovered individuals R in (1.2) since $S(t,\omega)+I(t,\omega)+R(t,\omega) = N(\omega)$ and $R(t,\omega)$ can be obtained from this equation almost surely. Thus, in our study, we will focus on the following stochastic system:

$$\begin{aligned} dS &= [bN - Sf(B) - bS]dt - \tau_1\frac{SI}{N}dW_1 + \tau_2\frac{SI}{N^2}(N-S)dW_2,\\ dI &= [Sf(B) - (\gamma+b)I]dt + \tau_1\frac{SI}{N}dW_1 - \tau_2\frac{SI^2}{N^2}dW_2,\\ dB &= (eI - mB)dt - \tau_3 B dW_3. \end{aligned} \tag{1.3}$$

The analysis of stochastic differential equation systems is difficult in general. It should be noticed that cholera epidemic models are SIR models coupled with infective cholera vibrios from the environment. Analysis of such models can borrow tools from SIR model analysis although there are extra difficulties as we did in [23] and other papers about cholera models. To conduct an analysis of the model (1.3),

it is nature to study relevant SIR models in terms of stochastic differential equations in the literature. Lahrouz *et al.* conducted a study of a stochastic SIRS epidemic model where the white noise is 1-dimensional [15]. In their second paper [14], they studied a similar model with 4-dimensional white noise. Yang *et al.* studied the ergodicity of two special epidemic models with 3 or 4-dimensional white noise [25]. Recently, Bao *et al.* conducted a study about the stationary distribution of a stochastic SIRS epidemic model where they provided a stochastic basic production which can characterize the dynamical behaviors of their model [3]. The major analytical tool in these studies is stochastic versions of Lyapunov functions. The advantage is that these models all have specific and explicit incidence functions which allow to construct a specific stochastic Lyapunov function. In our proposed model above, the incidence function is not specific given. This creates an extra difficulty in our study. For example, to circumvent this difficulty in the proof of existence of the stationary ergodic distribution, we construct several stochastic Lyapunov functions and combine them together to form the final stochastic Lyapunov function. If we take noise intensities to be zeros in the model (1.3), our proof gives a new proof of [23, Theorem 2.10] which was proved by using the theory of monotone dynamical systems. We define the basic stochastic reproduction value which is a sum of the basic reproduction number for the deterministic model and the function of noise intensities. From the formula of the basic stochastic reproduction value, we can also easily deduce that the noises can suppress the outbreak of cholera infections.

The rest of the paper is organized as follows. In Sec. 2, we present a basic analysis including existence and uniqueness of positive global solutions, and the condition under which the cholera epidemic will extinct or persist. In Sec. 3, we prove the existence of a unique stationary ergodic distribution. In Sec. 4, we carry out numerical simulations for three cases in both stochastic model and its corresponding deterministic model. The paper closes with Sec. 5.

2. Basic Dynamics

In this section, we discuss the almost surely positive invariant domain for the system (1.3) in which the solution will exist uniquely and globally for any initial value. We define the basic stochastic reproduction value $\mathcal{R}_s$, and point out its epidemic implication. We show the cholera infection will die out if $\mathcal{R}_s < 1$ almost surely, while the cholera epidemic will persist almost surely when $\mathcal{R}_s > 1$.

Throughout this paper, we specify an appropriate filtered probability space as follows. Let $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^3), \omega(0) = 0\}$, $\mathcal{F}$ be the Borel σ-algebra on Ω and $\mathbb{P}$ be the measure induced by $\{W_t\}_{t\in\mathbb{R}}$, a two-sided 3-dimensional Wiener process. The elements of Ω can be identified with paths of the Wiener process $\omega(t) = W_t(\omega)$. We consider the $\mathbb{P}$-completion $\mathcal{F}^{\mathbb{P}}$ of $\mathcal{F}$, i.e. $\mathcal{F}^{\mathbb{P}}$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. The filtration $\mathcal{F}_t^{\mathbb{P}}$ is given by the canonical filtration generated by the Wiener process W_t completed by all $\mathbb{P}$-null sets of $\mathcal{F}^{\mathbb{P}}$. Denote the probability measure given by the

extension of the probability measure $\mathbb{P}$ to $\mathcal{F}^{\mathbb{P}}$ again by $\mathbb{P}$. Therefore, a $\mathbb{P}$-completed filtered probability space $(\Omega, \mathcal{F}^{\mathbb{P}}, \{\mathcal{F}_t^{\mathbb{P}}\}_{t\in\mathbb{R}}, \mathbb{P})$ is obtained.

For convenience, we write our stochastic differential equation system (1.3) in the vector form. Let $X = (S, I, B)^T$,

$$F(X) = \begin{bmatrix} bN - Sf(B) - bS \\ Sf(B) - (\gamma + b)I \\ eI - mB \end{bmatrix}, \quad \text{and} \quad \sigma(X) = \begin{bmatrix} -\tau_1 \dfrac{SI}{N} & \tau_2 \dfrac{SI}{N^2}(N - S) & 0 \\ \tau_1 \dfrac{SI}{N} & -\tau_2 \dfrac{SI^2}{N^2} & 0 \\ 0 & 0 & -\tau_3 B \end{bmatrix}.$$

Then (1.2) can be written as

$$dX = F(X)\, dt + \sigma(X)\, dW_t, \tag{2.1}$$

where $W_t = (W_1, W_2, W_3)^T$ is the 3-dimensional Wiener process defined above.

We define the diffusion matrix for (2.1) as $A(X) = \sigma(X)\sigma(X)^T = (a_{ij}(X))$. We also define the differential operator L associated with (2.1) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{3} F_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} a_{ij}(X) \frac{\partial^2}{\partial x_i \partial x_j}.$$

The incidence function $f(B)$, depending on the concentration B of cholera vibrios in the environmental water resources, determines the rate of new infection introduced into the population and this function includes multiple transmission pathways. Note that $f(0) = 0$, $f \geq 0$, $f' \geq 0$, and $f'' \leq 0$ [23]. For example, $f(B) = \beta \frac{B}{K+B}$ with β being the contact rate with the contaminated water, and K the pathogen concentration that yields 50% chance of catching cholera infection.

2.1. *Existence and uniqueness of global and positive solutions*

In the stochastic differential equation setting, our first concern is whether the solution exists globally, and it is important to know where the solution lives in. In order for a general SDE to have a unique global solution (i.e. no explosion within a finite time) for any given initial value, the drift and diffusion coefficients of the equation are generally required to satisfy the linear growth condition and the Lipschitz condition (see [9, 12]). Unfortunately, the coefficient F of (2.1) does not satisfy the linear growth condition as the incidence function is nonlinear, so the solution of (1.3) might explode at a finite time (see [1]). In this subsection, using [12, Corollary 3.1], we will show that the solution of (1.3) exists uniquely and globally, and prove that

$$\Delta = \{(S, I, B) \in \mathbb{R}_+^3 : S + I < N\}$$

is almost surely positively invariant domain for the system (1.3), where

$$\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}.$$

Theorem 2.1. *For any initial value* $(S(0), I(0), B(0)) \in \Delta$, *the system* (1.3) *admits a unique solution* $(S(t), I(t), B(t))$ *on* $t \geq 0$, *and this solution remains in* Δ *almost surely.*

Proof. Let $\Delta_n = \{(S, I, B)^T \in \Delta : S > \frac{1}{n}, I > \frac{1}{n}, \frac{1}{n} < B < n, S + I < N - \frac{1}{n}\}$, then $\Delta_n \subset \Delta_{n+1}$ and $\overline{\Delta}_n \subseteq \Delta$. Since f is continuous and B is bounded on each Δ_n, $f(B)$ is bounded on each Δ_n. Hence, the coefficients F and σ satisfy the Lipschitz condition and the linear growth condition on each Δ_n. Now, we define a function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$V(S, I, B) = -\log \frac{S}{N} - \log \frac{I}{N} + \frac{f'(0)}{m}(B + 1 - \log B).$$

Since $x - 1 - \log x \geq 0$ for all $x > 0$, it is obvious that V is non-negative and twice continuously differentiable in Δ. Applying Ito's formula, we compute

$$\begin{aligned} LV &= -\frac{bN}{S} + f(B) + b - \frac{Sf(B)}{I} + \gamma + b + \frac{f'(0)e}{m} I - f'(0)B - \frac{f'(0)e}{m}\frac{I}{B} + f'(0) \\ &\quad + \frac{\tau_1^2}{2}\left(\frac{S^2}{N^2} + \frac{I^2}{N^2}\right) + \frac{\tau_2^2}{2}\left(\frac{I^2(N-S)^2}{N^4} + \frac{S^2 I^2}{N^4}\right) + \frac{\tau_3^2 f'(0)}{2m} \\ &\leq 2b + \gamma + f'(0) + \tau_1^2 + \tau_2^2 + \frac{\tau_3^2 f'(0)}{2m} =: \kappa. \end{aligned}$$

Here, we have used the equality $f(B) \leq f'(0)B$, which is followed from the fact that $\frac{f(B)}{B} = \frac{f(B)-f(0)}{B-0} = f'(\eta)$ for some $0 < \eta < B$ and, since f' is decreasing, $f'(\eta) \leq f'(0)$. Notice that we can assume $f'(0) > 0$ because, otherwise, $f(B) \leq 0$ for all $B > 0$ and, since $f \geq 0$, $f(B) \equiv 0$. Now, since $x - 1 - \log x \geq 0$ for all $x > 0$, $V \geq \frac{2f'(0)}{m}$. Therefore $LV \leq \bar{c}V$ where $\bar{c} = \frac{\kappa m}{2f'(0)}$.

Next, we would like to show that, when $X = (S, I, B) \in \Delta \backslash \Delta_n$, $V(X)$ approaches ∞ as $n \to \infty$. Suppose the solution $X = (S, I, B)$ of the system (1.3) is in $\Delta \backslash \Delta_n$. As $n \to \infty$, the solution $X = (S, I, B)$ will reach the boundary, $\partial\Delta$, of the positive domain Δ. Note that the boundary $\partial\Delta$ includes 5 faces $S = 0$, $I = 0$, $B = 0$, $B = \infty$, and $S + I = N$. Clearly, $-\log \frac{S}{N} \to \infty$ as $S \to 0$, $-\log \frac{I}{N} \to \infty$ as $I \to 0$, and $B + 1 - \log B \to \infty$ as either $B \to 0$ or $B \to \infty$. When $S + I = N$, adding the first two equations of the system (1.3) yields

$$d(S + I) = [b(N - S - I) - \gamma I]dt + \tau_2 \frac{SI}{N^2}(N - S - I)dW_2,$$

which implies that

$$\frac{dN}{dt} = \frac{d}{dt}(S + I) = -\gamma I.$$

Since N is constant a.s., $I = 0$ and hence $V(X) = \infty$. Thus the function $V(X)$ will blow up to ∞ when $X = (S, I, B)$ approaches $\partial\Delta$. This completes the proof. □

2.2. *Extinction of cholera infections*

Recall that, in [23], the dynamical behavior of the deterministic part of the system (1.3) is determined by the basic reproduction number $R_0 = \frac{N}{\gamma+b} f'(0)\frac{e}{m}$. As $R_0 < 1$, there is only one disease-free equilibrium (DFE) $X_0 = (N, 0, 0)$ in Δ, which is globally asymptotically stable. When $R_0 > 1$, there are two non-negative equilibria, X_0 and the endemic equilibrium $X^* = (S^*, I^*, B^*)$. X_0 becomes unstable; while X^* is globally asymptotically stable, meaning that the deterministic system becomes uniformly persistent and any trajectory of the deterministic model starting at any point in Δ converges to X^*. It can be shown that X^* uniquely exists without explicit expression since the incidence function f is not explicitly given. However, for the stochastic system (1.3), we derive a sufficient condition for the extinction of cholera infections in terms of the basic reproduction number for the deterministic model, noise intensities, and system parameters, which we call the basic stochastic reproduction value. The basic stochastic reproduction value is given by

$$\mathcal{R}_s = R_0 - \frac{\delta}{m},$$

where $\delta := \delta(\tau_1, \tau_2, \tau_3)$ is the function of noise intensities with the property that, as (τ_1, τ_2, τ_3) approaches $(0, 0, 0)$, δ approaches 0 and, as one of the noise intensities increases, δ will be increasing. Specifically, we define a function of the solutions of the system as follows:

$$h(S, I, B) = \frac{\tau_1^2 S^2(I^2/N^2) + \tau_2^2(S^2 I^2/N^4)I^2 + \tau_3^2 \varpi^2 B^2}{2(I + \varpi B)^2},$$

where $\varpi := \frac{\gamma+b}{e}$. Here, $S := S(t)$, $I := I(t)$, and $B := B(t)$ are solution components of (1.3). Then, we define

$$\delta := \inf\{h(S, I, B) : (S, I, B) \text{ solves (1.3) with initial value in } \Delta\}.$$

Clearly, δ is a random variable which is a function of the noise intensities with the property that δ is increasing as one of the noise intensities increases and, as all the noises approach 0, $\delta \to 0$. We claim that $\delta > 0$. It suffices to show that, as the solution (S, I, B) approaches the boundary $\partial\Delta$, $h(S, I, B)$ is still strictly positive. On the face $S = 0$, the system (1.3) becomes

$$dI = -(\gamma + b)I dt,$$

$$dB = (eI - mB)dt - \tau_3 B dW_3,$$

and $h(0, I, B) = \frac{\tau_3^2 \varpi^2 B^2}{2(I+\varpi B)^2}$. If $I = 0$ then $h(0, 0, B) = \frac{\tau_3^2}{2} > 0$. Suppose $I > 0$. As B is close to 0, $\frac{dB}{dt} = eI > 0$ wp1 and hence B will be increasing. So, B cannot

be zero and thus $h(0, I, B) > 0$. When B is very large without bound, $h(0, I, B)$ approaches $\frac{\tau_3^2}{2} > 0$. Next, on the face $I = 0$, we get $h(S, 0, B) = \frac{\tau_3^2}{2} > 0$. Now, consider the dynamics of the system on the face $B = 0$. Then, (1.3) becomes

$$dS = (bN - bS)dt - \tau_1 \frac{SI}{N} dW_1 + \tau_2 \frac{SI}{N^2}(N - S)dW_2,$$

$$dI = -(\gamma + b)Idt + \tau_1 \frac{SI}{N} dW_1 - \tau_2 \frac{SI^2}{N^2} dW_2,$$

and $h(S, I, 0) = \frac{1}{2}(\tau_1^2 \frac{S^2}{N^2} + \tau_2^2 \frac{S^2 I^2}{N^4})$. When S is close to 0, the first equation becomes $\frac{dS}{dt} = bN > 0$ and so S is increasing. Hence S cannot be zero as time goes by. Thus $h(S, I, 0) > 0$. Finally, on the face $S + I = N$, adding the first two equations of (1.3) gives $\frac{d(S+I)}{dt} = -\gamma I$. But, since $N = S + I$ is constant a.s., so $I = 0$. Hence $h(S, I, B) = \frac{\tau_3^2}{2} > 0$. Since the solution (S, I, B) of (1.3) stays within Δ with probability 1 if it starts in Δ, all the above arguments involving δ are true with probability 1. Therefore, we have proved that

$$\delta = \inf_{(S,I,B)\in\Delta} h(S, I, B) > 0 \text{ w.p. 1.}$$

Theorem 2.2. *Let $(S(t), I(t), B(t))$ be the solution of the stochastic system* (1.3) *with the initial value $(S(0), I(0), B(0)) \in \Delta$. If $\mathcal{R}_s < 1$, then the solution $(S(t), I(t), B(t))$ converges to $X_0 = (N, 0, 0)$ almost surely.*

Proof. First, we apply Ito's formula for the function $\log(I(t) + \varpi B(t))$ as follows:

$$\begin{aligned}
d(\log(I + \varpi B)) &= \frac{1}{I + \varpi B}[Sf(B) - (\gamma + b)I]dt + \frac{\varpi}{I + \varpi B}(eI - mB)dt \\
&\quad - \frac{\tau_1^2 S^2 (I^2/N^2) + \tau_2^2 (S^2 I^2/N^4) I^2 + \tau_3^2 \varpi^2 B^2}{2(I + \varpi B)^2} dt \\
&\quad + \frac{\tau_1 (S/N) I dW_1}{I + \varpi B} - \frac{\tau_2 (SI/N^2) I dW_2}{I + \varpi B} - \frac{\tau_3 \varpi B dW_3}{I + \varpi B} \\
&= \left[\frac{Sf(B) - m\varpi B}{I + \varpi B} - h(S, I, B)\right] dt \\
&\quad + \frac{\tau_1 (S/N) I dW_1}{I + \varpi B} - \frac{\tau_2 (SI/N^2) I dW_2}{I + \varpi B} - \frac{\tau_3 \varpi B dW_3}{I + \varpi B} \\
&\leq \left[m(R_0 - 1)\frac{\varpi B}{I + \varpi B} - \delta\right] dt \\
&\quad + \frac{\tau_1 (S/N) I dW_1}{I + \varpi B} - \frac{\tau_2 (SI/N^2) I dW_2}{I + \varpi B} - \frac{\tau_3 \varpi B dW_3}{I + \varpi B}. \qquad (2.2)
\end{aligned}$$

Here, we have used the equalities $Sf(B) \leq Nf'(0)B$.

Next, integrating (2.2) from 0 to t and dividing by t, we get

$$\begin{aligned}
&\frac{1}{t}\log(I(t)+\varpi B(t))\\
&\quad\le \frac{1}{t}\log(I(0)+\varpi B(0)) + m(R_0-1)\frac{1}{t}\int_0^t \frac{\varpi B(s)}{I(s)+\varpi B(s)}ds - \delta\\
&\quad\quad + \frac{\tau_1}{N}\frac{1}{t}\int_0^t \frac{S(s)I(s)dW_1(s)}{I(s)+\varpi B(s)} - \frac{\tau_2}{N^2}\frac{1}{t}\int_0^t \frac{S(s)I(s)^2dW_2(s)}{I(s)+\varpi B(s)}\\
&\quad\quad - \tau_3\frac{1}{t}\int_0^t \frac{\varpi B(s)dW_3(s)}{I(s)+\varpi B(s)}. \qquad (2.3)
\end{aligned}$$

By the strong law of large numbers for local martingales, since all the integrands of the last three stochastic integrals in the right-hand side of (2.3) are positive and bounded a.s., these stochastic integrals approach 0 almost surely as $t\to\infty$.

Now, we separately consider two cases, $R_0>1$ and $R_0\le 1$. When $R_0>1$, then

$$\begin{aligned}
\frac{1}{t}\log(I(t)+\varpi B(t)) &\le \frac{1}{t}\log(I(0)+\varpi B(0)) + m\left(R_0-1-\frac{\delta}{m}\right)\\
&\quad + \frac{\tau_1}{N}\frac{1}{t}\int_0^t \frac{S(s)I(s)dW_1(s)}{I(s)+\varpi B(s)} - \frac{\tau_2}{N^2}\frac{1}{t}\int_0^t \frac{S(s)I(s)^2dW_2(s)}{I(s)+\varpi B(s)}\\
&\quad - \tau_3\frac{1}{t}\int_0^t \frac{\varpi B(s)dW_3(s)}{I(s)+\varpi B(s)}.
\end{aligned}$$

Letting $t\to\infty$ yields

$$\limsup_{t\to\infty}\frac{\log(I(t)+\varpi B(t))}{t} \le m(\mathcal{R}_s-1) < 0.$$

This means that $(I(t),B(t))$ approaches $(0,0)$ exponentially almost surely.

In order to show that $\lim_{t\to\infty}S(t)=N$ almost surely when $R_0>1$, we need to utilize the non-negative semi-martingale convergence theorem (see [16, Theorem 3.9 on p. 14]). From the first equation of (1.2), we can write

$$\begin{aligned}
N-S(t) &= N-S(0)+\int_0^t S(s)f(B(s))ds - \int_0^t b(N-S(s))ds\\
&\quad + \frac{\tau_1}{N}\int_0^t S(s)I(s)dW_1(s) - \frac{\tau_2}{N^2}\int_0^t S(s)I(s)(N-S(s))dW_2(s).
\end{aligned}$$

Denote

$$A_t := \int_0^t S(s)f(B(s))ds, \quad U_t := \int_0^t b(N-S(s))ds,$$

$$M_t := \frac{\tau_1}{N}\int_0^t S(s)I(s)dW_1(s) - \frac{\tau_2}{N^2}\int_0^t S(s)I(s)(N-S(s))dW_2(s).$$

Notice that A_t and U_t are two continuous adapted (i.e. $\mathcal{F}_t$-measurable) increasing processes with $A_0 = U_0 = 0$. Since stochastic integrals with respect to Brownian Motions are martingales, hence local martingales, and sum of martingales are again a martingale, M_t is a local martingale with $M_0 = 0$. Clearly, $N - S(t) > 0$ a.s. On the other hand, by the proof above, $\limsup_{t\to\infty} \frac{\log B(t)}{t} \le m(\mathcal{R}_s - 1)$. Then, there exists a $\Theta > 0$ such that $t > \Theta$ implies $B(t) \le \exp\{m(\mathcal{R}_s - 1)t\}$. Since $f(B(t)) \le f'(0)B(t)$, for $t > \Theta$

$$\begin{aligned} A_t &= \int_0^\Theta S(s)f(B(s))ds + \int_\Theta^t S(s)f(B(s))ds \\ &\le \int_0^\Theta S(s)f(B(s))ds + Nf'(0)\int_\Theta^t B(s)ds \\ &\le \int_0^\Theta S(s)f(B(s))ds + \frac{Nf'(0)}{m(1-\mathcal{R}_s)}(\exp\{m(\mathcal{R}_s - 1)\Theta\} - \exp\{m(\mathcal{R}_s - 1)t\}), \end{aligned}$$

which implies that $\lim_{t\to\infty} A_t < \infty$ a.s. By [16, Theorem 3.9 on p. 14], $\lim_{t\to\infty}(N - S(t)) < \infty$ a.s. and $\lim_{t\to\infty}\int_0^t (N - S(s))ds < \infty$ a.s.

If $S(t)$ did not converge a.s. to N, then there would be an $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) > 0$ so that $\liminf_{t\to\infty}(N - S(t)) > 0$ for all $\omega \in \Omega_1$. Fix $\omega \in \Omega_1$, then $\liminf_{t\to\infty}(N - S(t,\omega)) = p(\omega) > 0$. So, there exists a $T := T(\omega) > 0$ so that $t \ge T$ implies $N - S(t,\omega) > \frac{1}{2}p(\omega)$. Hence,

$$\begin{aligned} \int_0^\infty (N - S(s,\omega))ds &= \int_0^T (N - S(s,\omega))ds + \int_T^\infty (N - S(s,\omega))ds \\ &\ge \int_T^\infty (N - S(s,\omega))ds \ge \int_T^\infty \frac{1}{2}p(\omega)ds = \infty. \end{aligned}$$

This follows that $\Omega_1 \subset \Omega_2$, where $\Omega_2 = \{\omega \in \Omega : \int_0^\infty (N - S(s,\omega))ds = \infty\}$. But then $\mathbb{P}(\Omega_2) > 0$, which contradicts the fact $\int_0^\infty (N - S(s))ds < \infty$ a.s. Therefore

$$\lim_{t\to\infty}(N - S(t)) = 0 \quad \text{a.s.}$$

Therefore, we have proved Theorem 2.2 for the case $R_0 > 1$.

If $R_0 \le 1$ then, of course, $\mathcal{R}_s < 1$ and the inequality (2.3) implies

$$\begin{aligned} \frac{1}{t}\log(I(t) + \varpi B(t)) \le \frac{1}{t}\log(I(0) + \varpi B(0)) - \delta + \frac{\tau_1}{N}\frac{1}{t}\int_0^t \frac{S(s)I(s)dW_1(s)}{I(s) + \varpi B(s)} \\ - \frac{\tau_2}{N^2}\frac{1}{t}\int_0^t \frac{S(s)I(s)^2 dW_2(s)}{I(s) + \varpi B(s)} - \tau_3\frac{1}{t}\int_0^t \frac{\varpi B(s)dW_3(s)}{I(s) + \varpi B(s)}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\limsup_{t\to\infty} \frac{\log(I(t) + \varpi B(t))}{t} \le -\delta < 0.$$

Thus, by the same argument as above, $(S(t), I(t), B(t)) \to (N, 0, 0)$ almost surely. This completes the proof. □

Remark. It may happen that $R_0 > 1$ but $\mathcal{R}_s < 1$. This means that the cholera infection in the stochastic model (1.3) dies out while the cholera infection in the deterministic part of (1.3) is persistent. Thus, theoretically, white noises can suppress the outbreak of cholera infections. In reality, it may happen that we estimate parameter values and use the deterministic model to predict the cholera infection will spread out, however, the cholera infection may just die out. We will perform numerical simulations to demonstrate this case in Sec. 4. This may remind us we should consider environmental fluctuations.

2.3. *Persistence of cholera infections*

The next theorem will give a condition for the persistence of cholera infections in the stochastic model (1.3).

Theorem 2.3. *Let $(S(t), I(t), B(t))$ be the solution of the stochastic system* (1.3) *with the initial value $(S(0), I(0), B(0)) \in \Delta$. If $\mathcal{R}_s > 1$, then the solution of the stochastic model* (1.3) *will satisfy*

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t (N - S(s))ds > 0,$$

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t I(s)ds > 0, \quad \textit{and} \quad \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t B(s)ds > 0.$$

So, we can conclude that the solution $(S(t), I(t), B(t))$ does not converge to the disease-free equilibrium $(N, 0, 0)$ with positive probability.

Proof. Denote

$$\Omega_3 = \left\{\omega \in \Omega : \lim_{t\to\infty} S(t, \omega) = N\right\} \quad \text{and} \quad \Omega_4 = \left\{\omega \in \Omega : \lim_{t\to\infty} B(t, \omega) = 0\right\}.$$

Assume that $\mathbb{P}(\Omega_3^c \cup \Omega_4^c) > 0$. Then either $\mathbb{P}(\Omega_3^c) > 0$ or $\mathbb{P}(\Omega_4^c) > 0$. If $\mathbb{P}(\Omega_3^c) > 0$ then, for $\omega \in \Omega_3^c$, $\liminf_{t\to\infty}(N - S(t, \omega)) > 0$. Hence

$$\mathbb{E}\liminf_{t\to\infty}(N - S(t)) \geq \mathbb{E}\liminf_{t\to\infty}(N - S(t))\mathbb{1}_{\Omega_3^c} > 0.$$

Thus, by general L'Hospital Rule,

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t (N - S(s))ds \geq \mathbb{E}\liminf_{t\to\infty}(N - S(t, \omega)) > 0.$$

Next, adding the first two equations of (1.2), we get

$$d(S + I) = [b(N - S) - (\gamma + b)I]dt + \tau_2 \frac{SI}{N^2}(N - S - I)dW_2.$$

Integrating this from 0 to t and then dividing by t gives

$$(\gamma+b)\frac{1}{t}\int_0^t I(s)ds = -\frac{S(t)+I(t)-S(0)-I(0)}{t} + \frac{1}{t}\int_0^t b(N-S(s))ds + \tau_2\frac{1}{t}\int_0^t (S(s)I(s)/N^2)(N-S(s)-I(s))dW_2(s).$$

Since $(S(t)I(t)/N^2)(N-S(t)-I(t))$ is bounded a.s., by strong law of large numbers for local martingales

$$\liminf_{t\to\infty} \tau_2\frac{1}{t}\int_0^t (S(s)I(s)/N^2)(N-S(s)-I(s))dW_2(s) = 0 \quad \text{a.s.}$$

By Theorem 2.1, $S(t)+I(t) < N$ with probability 1 and so $\frac{S(t)+I(t)-S(0)-I(0)}{t}$ converges a.s. 0 as $t\to\infty$. Then letting $t\to\infty$, we have almost surely,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds = \frac{b}{\gamma+b}\liminf_{t\to\infty}\frac{1}{t}\int_0^t (N-S(s))ds.$$

Thus

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t I(s)ds = \frac{b}{\gamma+b}\liminf_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t (N-S(s))ds > 0.$$

On the other hand, from the third equation of (1.2), we get

$$m\frac{1}{t}\int_0^t B(s)ds = -\frac{B(t)-B(0)}{t} + e\frac{1}{t}\int_0^t I(s)ds - \tau_3\frac{1}{t}\int_0^t B(s)dW_3(s).$$

Taking expectation both sides and letting $t\to\infty$, we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t B(s)ds = \frac{e}{m}\liminf_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t I(s)ds > 0.$$

If $P(\Omega_4^c) > 0$ then the same argument as above implies

$$\mathbb{E}\liminf_{t\to\infty} B(t) > 0.$$

Using the general L'Hospital Rule gives

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t B(s)ds \geq \mathbb{E}\liminf_{t\to\infty} B(t) > 0.$$

By the above proof, we get

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t I(s)ds = \frac{m}{e}\liminf_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t B(s)ds > 0$$

and

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t (N-S(s))ds = \frac{\gamma+b}{b}\liminf_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t I(s)ds > 0.$$

Now, we suppose that $\mathbb{P}(\Omega_3 \cap \Omega_4) = 1$. This follows that $\mathbb{P}(\Omega_3) = 1$ and $\mathbb{P}(\Omega_4) = 1$. Given $\epsilon > 0$, since $\lim_{x\downarrow 0}(f(x) - f'(0)x) = 0$, there is a $\delta > 0$ such that $0 < x < \delta$, which implies $f(x) > f'(0)x - \epsilon$. For almost sure, there exists a $T > 0$ such that $t \geq T$ implies $N - \delta < S(t) < N$ and $0 < B(t) < \delta$. Now, from the second and third equations of (1.3),

$$d(I + \varpi B) = (Sf(B) - m\varpi B)dt + \tau_1 \frac{SI}{N} dW_1 - \tau_2 \frac{SI^2}{N^2} dW_2 - \tau_3 \varpi B dW_3.$$

Then taking expectation both sides, we have

$$\frac{d}{dt}\mathbb{E}(I(t) + \varpi B(t)) = \mathbb{E}[S(t)f(B(t)) - m\varpi B(t)].$$

So, for $t \geq T$, $\mathbb{E}[S(t)f(B(t)) - m\varpi B(t)] > m\varpi(R_0 - 1)\mathbb{E}B(t) - \epsilon[N + f'(0)\mathbb{E}B(t)]$. Note that $0 < \mathbb{E}B(t) \leq \frac{\epsilon N}{m}$ for all $t \geq T$ and $R_0 > 1$ because $\mathcal{R}_s > 1$. Then, we can choose $\epsilon > 0$ sufficiently small so that $\frac{d}{dt}\mathbb{E}(I(t) + \varpi B(t)) > 0$ for $t \geq T$. It follows that $\liminf_{t\to\infty} \mathbb{E}I(t) > 0$. This completes the proof. □

3. Existence of a Stationary Ergodic Distribution

When $\mathcal{R}_s > 1$, then $R_0 > 1$, and the endemic equilibrium X^* for the deterministic model (1.1) is globally asymptotically stable [23]. We know that the endemic equilibrium X^* does not solve the stochastic model (1.3). However, in the section, we will show there exists a unique asymptotically stationary distribution for the solutions of the stochastic system (1.3) when $\mathcal{R}_s > 1$ and noise intensities are small and are bounded by model parameter quantities, which in turn implies the global stability in the stochastic sense.

Precisely, let us denote the probability measure induced by the solution $X(t) = (S(t), I(t), B(t))$ with initial value x by $P(t, x, \cdot)$, that is,

$$P(t, x, B) = \mathbb{P}\{X(t) \in B \,|\, X(0) = x\}, \quad \text{for any Borel set } B \subseteq \Delta.$$

If there is a probability measure $\mu(\cdot)$ on the measurable space $(\Delta; \mathcal{B}(\Delta))$ such that for any $x \in \Delta$ we have $\lim_{t\to\infty} P(t, x, B) = \mu(B)$, then we say that the stochastic system (1.3) has a stationary distribution μ.

Based on Khasminskii [12], if there exists a bounded open domain $U \subset \Delta$ with a regular boundary such that its closure $\bar{U} \subset \Delta$, and the following conditions hold:

(i) In the open domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from 0.
(ii) If $x \in \Delta \backslash U$, the mean time τ at which a solution path starting from x reaches the set U is finite, and $\sup_{x\in K} \mathbb{E}_x \tau < \infty$ for every compact subset $K \subset \Delta$.

Then the solution process $X(t)$ of the system (2.1) has a unique stationary ergodic distribution μ. Moreover, if f is a function integrable with respect to the measure

μ, then for all $x \in \Delta$, we have

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X^x(s))ds = \int_\Delta f(x)\mu(dx)\right\} = 1.$$

Remark. To validate the assumption (i), it suffices to show that there exists a positive number c so that $\sum_{i,j=1}^3 a_{ij}(X)\xi_i\xi_j \geq c|\xi|^2$ for all $X \in U$ and $\xi \in \mathbb{R}^3$ (see [9] and Rayleigh's principle in [21]). For the assumption (ii), it is enough to find a non-negative C^2 function V such that, for any $X \in \Delta\backslash U$, $LV(X) \leq -C$ for some constant $C > 0$ (see [26]).

As we mention that when $\mathcal{R}_s = R_0 - \frac{\delta}{m} > 1$, then $R_0 > 1$. Hence, there is a unique endemic equilibrium $X^* = (S^*, I^*, B^*)$ for the deterministic part of the model (1.3). The conditions of equilibriums give the following identities which are needed in our proof:

$$bN = S^* f(B^*) + bS^*, \quad S^* f(B^*) = (\gamma + b)I^*, \quad eI^* = mB^*. \tag{3.1}$$

Now, we state and prove our main theorem.

Theorem 3.1. *Let $(S(t), I(t), B(t))$ be the solution of the stochastic system* (1.3) *with any initial value $(S(0), I(0), B(0)) \in \Delta$. Assume that $\mathcal{R}_s > 1$, $\tau_3 < \sqrt{2m}$, and*

$$0 < \rho < \min\left\{m_1 S^{*2}, m_2 I^{*2}, m_3 B^{*2}, \left(\frac{N - S^* - I^*}{\frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_2}}}\right)^2\right\}, \tag{3.2}$$

where

$$\rho = \tau_1^2\theta_1 + \tau_2^2\theta_2 + \tau_3^2\theta_3, \quad m_1 = b, \quad m_2 = \frac{\gamma + b}{4}, \quad m_3 = \frac{m^2\varpi}{4e},$$

$$\theta_1 = \left[N^2 + S^*I^* + \frac{S^*(S^* + I^*)f(B^*)}{b}\right]\frac{(\gamma + 2b)^2}{4b(\gamma + b)},$$

$$\theta_2 = \left[2N^2 + S^*I^* + \frac{S^*(S^* + I^*)f(B^*)}{b}\right]\frac{(\gamma + 2b)^2}{4b(\gamma + b)} + \frac{5N^2}{2},$$

$$\theta_3 = \frac{(\gamma + 2b)^2}{4b(\gamma + b)}(\varpi S^*B^* + \varpi S^*B^* f(B^*)) + \frac{m\varpi}{2e}M^2, \quad \varpi := \frac{\gamma + b}{e}.$$

Then there is a unique stationary distribution μ for the stochastic system (1.3) *and the solution is ergodic with respect to μ.*

Proof. First, we define the positive C^2 function $V_1 : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$V_1(S, I, B) = S - S^* - S^*\log\frac{S}{S^*} + I - I^* - I^*\log\frac{I}{I^*}$$
$$+ \varpi\left(B - B^* - B^*\log\frac{B}{B^*}\right).$$

Applying Ito's formula for this function and using (3.1), we get

$$\begin{aligned} LV_1 &= 3S^*f(B^*) - S^*f(B^*)\left[\frac{S^*}{S} + \frac{I}{I^*}\frac{B^*}{B} + \frac{f(B)}{f(B^*)}\frac{S}{S^*}\frac{I^*}{I}\right] \\ &\quad + S^*f(B^*)\left[\frac{f(B)}{f(B^*)} - \frac{B}{B^*}\right] + bS^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \\ &\quad + \frac{\tau_1^2}{2}\left(S^*\frac{I^2}{N^2} + I^*\frac{S^2}{N^2}\right) + \frac{\tau_2^2}{2}\left(S^*\frac{I^2}{N^2}\frac{(N-S)^2}{N^2} + I^*\frac{S^2I^2}{N^4}\right) + \frac{\tau_3^2}{2}\varpi B^*. \end{aligned}$$

By Cauchy's inequality,

$$\frac{S^*}{S} + \frac{I}{I^*}\frac{B^*}{B} + \frac{f(B)}{f(B^*)}\frac{S}{S^*}\frac{I^*}{I} \geq 4 - \frac{f(B^*)B}{f(B)B^*}.$$

Then, note that $\frac{S}{N} \leq 1$, $\frac{I}{N} \leq 1$, and $\frac{N-S}{N} \leq 1$,

$$\begin{aligned} LV_1 &\leq S^*f(B^*)\left[\frac{f(B^*)B}{f(B)B^*} + \frac{f(B)}{f(B^*)} - \frac{B}{B^*} - 1\right] + bS^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \\ &\quad + \frac{1}{2}(\tau_1^2 + \tau_2^2)(S^* + I^*) + \frac{\tau_3^2}{2}\varpi B^*. \end{aligned}$$

Since $f(B)$ and $\frac{B}{f(B)}$ are both increasing,

$$\frac{f(B^*)B}{f(B)B^*} + \frac{f(B)}{f(B^*)} - \frac{B}{B^*} - 1 = \frac{1}{B^*}(f(B) - f(B^*))\left[\frac{B^*}{f(B^*)} - \frac{B}{f(B)}\right] \leq 0.$$

This implies that

$$LV_1 \leq bS^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \frac{1}{2}(\tau_1^2 + \tau_2^2)(S^* + I^*) + \frac{\tau_3^2}{2}\varpi B^*. \tag{3.3}$$

Secondly, we define the positive C^2 function $V_2 : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$V_2(I, B) = I - I^* - I^*\log\frac{I}{I^*} + \varpi\left(B - B^* - B^*\log\frac{B}{B^*}\right).$$

By computation and using (3.1),

$$\begin{aligned} LV_2 &= (S - S^*)(f(B) - f(B^*)) + Sf(B^*) + S^*f(B) - S^*f(B^*)\frac{B}{B^*} \\ &\quad + S^*f(B^*)\left[1 - \frac{S}{S^*}\frac{f(B)}{f(B^*)}\frac{I^*}{I} - \frac{I}{I^*}\frac{B^*}{B}\right] \\ &\quad + \frac{1}{2}\left(\tau_1^2I^*\frac{S^2}{N^2} + \tau_2^2I^*\frac{S^2I^2}{N^4}\right) + \frac{1}{2}\tau_3^2\varpi B^*. \end{aligned}$$

Again by Cauchy's inequality, since

$$\frac{S}{S^*}\frac{B}{B^*}\frac{f(B)B^*}{f(B^*)B}\frac{I^*}{I}+\frac{I}{I^*}\frac{B^*}{B}+\frac{S^*}{S}+\frac{f(B^*)B}{f(B)B^*}\geq 4,$$

it implies that

$$1-\frac{S}{S^*}\frac{f(B)}{f(B^*)}\frac{I^*}{I}-\frac{I}{I^*}\frac{B^*}{B}\leq\frac{S^*}{S}-2+\frac{f(B^*)B}{f(B)B^*}-1.$$

Thus, note that $\frac{S^2}{N^2}\leq 1$ and $\frac{S^2I^2}{N^4}\leq 1$,

$$\begin{aligned}LV_2&\leq(S-S^*)(f(B)-f(B^*))+S^*f(B^*)\left[\frac{S}{S^*}+\frac{S^*}{S}-2\right]\\&\quad+S^*f(B^*)\left[\frac{f(B)}{f(B^*)}-\frac{B}{B^*}+\frac{f(B^*)B}{f(B)B^*}-1\right]+\frac{1}{2}(\tau_1^2+\tau_2^2)I^*+\frac{\tau_3^2}{2}\varpi B^*.\end{aligned}$$

By the above proof, since $\frac{f(B)}{f(B^*)}-\frac{B}{B^*}+\frac{f(B^*)B}{f(B)B^*}-1\leq 0$,

$$\begin{aligned}LV_2&\leq(S-S^*)(f(B)-f(B^*))\\&\quad+S^*f(B^*)\left[\frac{S}{S^*}+\frac{S^*}{S}-2\right]+\frac{1}{2}(\tau_1^2+\tau_2^2)I^*+\frac{\tau_3^2}{2}\varpi B^*.\end{aligned}\tag{3.4}$$

Thirdly, we define the non-negative C^2 function $V_3:\mathbb{R}_+\to\mathbb{R}_+$ by $V_3(S)=\frac{1}{2}(S-S^*)^2$. Then

$$\begin{aligned}LV_3&=-b(S-S^*)^2-f(B)(S-S^*)^2-S^*(S-S^*)(f(B)-f(B^*))\\&\quad+\frac{1}{2}\tau_1^2S^2\frac{I^2}{N^2}+\frac{1}{2}\tau_2^2\frac{S^2I^2}{N^4}(N-S)^2\\&\leq-b(S-S^*)^2-S^*(S-S^*)(f(B)-f(B^*))+\frac{\tau_1^2}{2}S^2+\frac{\tau_2^2}{2}(N-S)^2.\end{aligned}\tag{3.5}$$

Combining (3.3)–(3.5) gives

$$\begin{aligned}&LV_3+S^*LV_2+\frac{S^*f(B^*)}{b}LV_1\\&\quad\leq-b(S-S^*)^2+\frac{\tau_1^2}{2}\left(S^2+S^*I^*+\frac{S^*(S^*+I^*)f(B^*)}{b}\right)\\&\qquad+\frac{\tau_2^2}{2}\left((N-S)^2+S^*I^*+\frac{S^*(S^*+I^*)f(B^*)}{b}\right)\\&\qquad+\frac{\tau_3^2}{2}(\varpi S^*B^*+\varpi S^*B^*f(B^*)).\end{aligned}$$

Fourthly, we define the non-negative C^2 functions $V_4 : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $V_5 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$V_4(S, I) = \frac{1}{2}(S - S^* + I - I^*)^2, \quad V_5(B) = \frac{1}{2}(B - B^*)^2.$$

Applying Ito's formula for these functions and using (3.1), we get

$$\begin{aligned}
LV_4 &= -b(S - S^*)^2 - \frac{\gamma + b}{2}(I - I^*)^2 - \frac{\gamma + b}{2}\left[I - I^* + \frac{\gamma + 2b}{\gamma + b}(S - S^*)\right]^2 \\
&\quad + \frac{(\gamma + 2b)^2}{2(\gamma + b)}(S - S^*)^2 + \frac{\tau_2^2}{2}\frac{S^2 I^2}{N^4}(N - S - I)^2 \\
&\le -b(S - S^*)^2 - \frac{\gamma + b}{2}(I - I^*)^2 + \frac{(\gamma + 2b)^2}{2(\gamma + b)}(S - S^*)^2 + \frac{\tau_2^2}{2}(N - S - I)^2, \\
LV_5 &= -\frac{m}{2}(B - B^*)^2 - \frac{m}{2}\left[B - B^* - \frac{e}{m}(I - I^*)\right]^2 + \frac{e^2}{2m}(I - I^*)^2 + \frac{\tau_3^2}{2}B^2 \\
&\le -\frac{m}{2}(B - B^*)^2 + \frac{e^2}{2m}(I - I^*)^2 + \frac{\tau_3^2}{2}B^2.
\end{aligned}$$

Then

$$\begin{aligned}
LV_4 + \frac{m(\gamma + b)}{2e^2}LV_5 &\le \frac{(\gamma + 2b)^2}{2(\gamma + b)}(S - S^*)^2 - b(S - S^*)^2 - \frac{\gamma + b}{4}(I - I^*)^2 \\
&\quad - \frac{m^2(\gamma + b)}{4e^2}(B - B^*)^2 + \frac{\tau_2^2}{2}(N - S - I)^2 \\
&\quad + \frac{m(\gamma + b)\tau_3^2}{4e^2}B^2.
\end{aligned}$$

Finally, we define the positive definite C^2 function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ so that

$$V := \frac{(\gamma + 2b)^2}{2b(\gamma + b)}\left[V_3 + S^* V_2 + \frac{S^* f(B^*)}{b}V_1\right] + V_4 + \frac{m(\gamma + b)}{2e^2}V_5.$$

Then, we have

$$\begin{aligned}
LV &\le -b(S - S^*)^2 - \frac{\gamma + b}{4}(I - I^*)^2 - \frac{m^2\varpi}{4e}(B - B^*)^2 \\
&\quad + \tau_1^2\left[S^2 + S^* I^* + \frac{S^*(S^* + I^*)f(B^*)}{b}\right]\frac{(\gamma + 2b)^2}{4b(\gamma + b)} \\
&\quad + \tau_2^2\left[(N - S)^2 + S^* I^* + \frac{S^*(S^* + I^*)f(B^*)}{b}\right]\frac{(\gamma + 2b)^2}{4b(\gamma + b)} \\
&\quad + \frac{\tau_2^2}{2}(N - S - I)^2 + \tau_3^2\left[\frac{(\gamma + 2b)^2}{4b(\gamma + b)}(\varpi S^* B^* + \varpi S^* B^* f(B^*)) + \frac{m\varpi}{4e}B^2\right].
\end{aligned}$$

From the third equation of (1.3), taking expectation both sides gives

$$\frac{d\mathbb{E}B}{dt} = e\mathbb{E}I - m\mathbb{E}B,$$

which implies that $\mathbb{E}B \leq \frac{eN}{m}$. By using Ito's formula, we have

$$dB^2 = 2B(eI - mB)dt + \tau_3^2 B^2 dt - 2\tau_3 B^2 dW_3.$$

Taking expectation both sides, we get

$$\frac{d\mathbb{E}B^2}{dt} = 2e\mathbb{E}(IB) - (2m - \tau_3^2)\mathbb{E}B^2.$$

Since $\mathbb{E}(IB) \leq N\mathbb{E}B \leq \frac{eN^2}{m}$,

$$\frac{d\mathbb{E}B^2}{dt} \leq \frac{2e^2N^2}{m} - (2m - \tau_3^2)\mathbb{E}B^2.$$

Suppose that $\tau_3^2 < 2m$. Then the above inequality implies that $\mathbb{E}B^2 \leq \frac{2e^2N^2}{m(2m-\tau_3^2)}$. So $B(t)$ is a Gaussian process with finite expectation and variance. Hence $B(t)$ must be bounded a.s. by some positive constant M. Note that $S^2 \leq N^2$, $(N-S)^2 \leq N^2 + S^2 \leq 2N^2$, and $(N-S-I)^2 \leq N^2 + S^2 + I^2 + 2SI \leq 5N^2$. Thus

$$LV \leq -m_1(S - S^*)^2 - m_2(I - I^*)^2 - m_3(B - B^*)^2 + \rho,$$

where m_1, m_2, m_3, and ρ are defined as in the statement of the theorem.

Finally, since $\rho \to 0$ as $(\tau_1, \tau_2, \tau_3) \to (0, 0, 0)$, we can choose τ_1, τ_2, and τ_3 small enough such that $\tau_3^2 < 2m$ and the condition (3.2) is fulfilled. Then the ellipsoid

$$-m_1(S - S^*)^2 - m_2(I - I^*)^2 - m_3(B - B^*)^2 + \rho = 0,$$

lies entirely in Δ. Take U to be any neighborhood of the ellipsoid with $\overline{U} \subset \Delta$ and thus for any $x \in \Delta \backslash U$ we get $LV \leq -C$ for some positive constant $C > 0$. Therefore, we finish the validation of the condition (ii). On the other hand, it is straightforward to see that the diffusion matrix $A(X)$ associated with (1.3) is uniformly elliptic in U, and so fulfills the condition (i). We then complete the proof. □

Remark. When the noise intensities $\tau_1 = 0$, $\tau_2 = 0$, and $\tau_3 = 0$, then this proof gives another proof, using Lyapunov functions, for the globally asymptotical stability of the endemic equilibrium X^* of the deterministic part of the model (1.3), which is different from the proof using monotone dynamical systems in [23].

4. Numerical Simulations

To illustrate our analytical results, we conduct some numerical simulations. To be concrete, we take the incidence function $f(B) = \beta\frac{B}{K+B}$ for our cholera model. We compare the solution behaviors of the stochastic model with that of its corresponding deterministic model. The derivative-free Milstein's strong order 1.0 method (see

[13, Chap. 11]) is used to approximate the strong solution of the stochastic system (1.2) with given initial value. The corresponding discretization equations are

$$\begin{cases} \tilde{S}_i = S_i + \left(bN - \beta\dfrac{S_iB_i}{K+B_i} - bS_i\right)\Delta t - \tau_1\dfrac{S_iI_i}{N}\sqrt{\Delta t} + \tau_2\dfrac{S_iI_i}{N^2}(N-S_i)\sqrt{\Delta t}, \\ \tilde{I}_i = I_i + \left(\beta\dfrac{S_iB_i}{K+B_i} - (\gamma+b)I_i\right)\Delta t + \tau_1\dfrac{S_iI_i}{N}\sqrt{\Delta t} - \tau_2\dfrac{S_iI_i^2}{N^2}\sqrt{\Delta t}, \\ \tilde{B}_i = B_i + (eI_i - mB_i)\Delta t - \tau_3B_i\sqrt{\Delta t}, \end{cases}$$

and

$$\begin{cases} S_{i+1} = S_i + \left(bN - \beta\dfrac{S_iB_i}{K+B_i} - bS_i\right)\Delta t - \tau_1\dfrac{S_iI_i}{N}\Delta W_i^1 + \tau_2\dfrac{S_iI_i}{N^2}(N-S_i)\Delta W_i^2 \\ \qquad + \dfrac{1}{2\sqrt{\Delta t}}\left\{\left(-\tau_1\dfrac{\tilde{S}_i\tilde{I}_i}{N} + \tau_1\dfrac{S_iI_i}{N}\right)[(\Delta W_i^1)^2 - \Delta t]\right. \\ \qquad \left. + \left(\tau_2\dfrac{\tilde{S}_i\tilde{I}_i}{N^2}(N-\tilde{S}_i) - \tau_2\dfrac{S_iI_i}{N^2}(N-S_i)\right)[(\Delta W_i^2)^2 - \Delta t]\right\}, \\ I_{i+1} = I_i + \left(\beta\dfrac{S_iB_i}{K+B_i} - (\gamma+b)I_i\right)\Delta t + \tau_1\dfrac{S_iI_i}{N}\Delta W_i^1 - \tau_2\dfrac{S_iI_i^2}{N^2}\Delta W_i^2 \\ \qquad + \dfrac{1}{2\sqrt{\Delta t}}\left\{\left(\tau_1\dfrac{\tilde{S}_i\tilde{I}_i}{N} - \tau_1\dfrac{S_iI_i}{N}\right)[(\Delta W_i^1)^2 - \Delta t]\right. \\ \qquad \left. + \left(-\tau_2\dfrac{\tilde{S}_i\tilde{I}_i^2}{N^2} + \tau_2\dfrac{S_iI_i^2}{N^2}\right)[(\Delta W_i^2)^2 - \Delta t]\right\}, \\ B_{i+1} = B_i + (eI_i - mB_i)\Delta t - \tau_3B_i\Delta W_i^3 \\ \qquad + \dfrac{1}{2\sqrt{\Delta t}}(-\tau_3\tilde{B}_i + \tau_3B_i)[(\Delta W_i^3)^2 - \Delta t], \end{cases}$$

where ΔW_i^1, ΔW_i^2, and ΔW_i^3 are $\sqrt{\Delta t}\,N(0,1)$-distributed independent random variables. The parameters are chosen as in Table 1 (see [7]). The unit for time is day, the unit for populations of human is one person, and for the concentration of V. cholerae is computed in the unit of 10^5 cells per mL.

Case 1. We demonstrate the situation where the cholera infection in both stochastic model and its corresponding deterministic model will die out when $\mathcal{R}_s < 1$ and $R_0 < 1$. Taking $\beta = 0.05$, $\tau_1 = \tau_2 = \tau_3 = 0.1$, and the values of other parameters in Table 1, we get $R_0 = 0.7937$ and $\mathcal{R}_s = 0.7853$. The initial values for both deterministic and stochastic models are the same (7,000, 1,000, 1.5). Figure 1 shows the solutions for both stochastic and its corresponding deterministic models, where the left column is for the deterministic model and the right column is for the

Table 1. Parameters and their values.

Parameters	Description	Values	Dimensions
N	Hypothetical total human population	10,000	persons
b	Natural human birth rate/death rate	0.001	persons/day
β	Rate of ingesting vibrios from the contaminated water	0.05–0.15	persons/day
γ	Rate of recovery from cholerae	0.02	person/day
e	Rate of contribution to V. cholerae	10	cells/mL/day/person
m	Net death rate of vibrios in the environment	0.3	cells/day
K	Concentration of V. cholerae in water that yields 50% chance of catching cholera	10^6	cells/mL

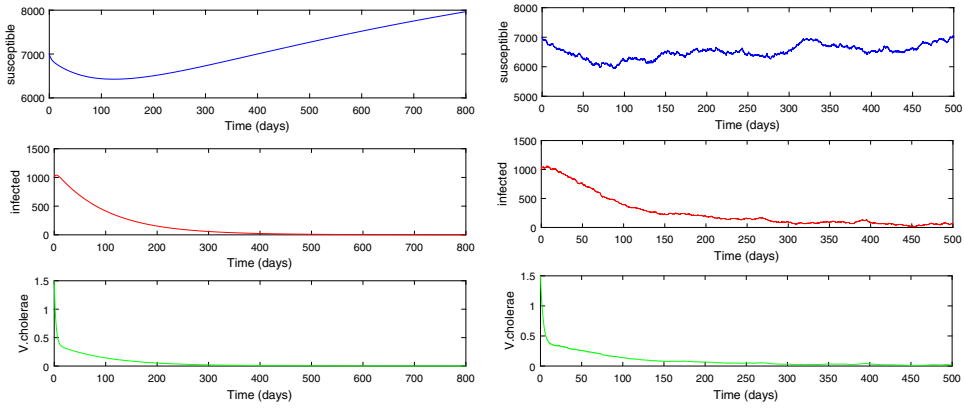

Fig. 1. Solution paths when $\mathcal{R}_s < 1$ and $R_0 < 1$. The right column is for the stochastic model while the left column is for its corresponding deterministic model.

stochastic model. These plots show the disease will die out in both deterministic and stochastic models. We also observe that the in stochastic model the disease dies out faster than that in its corresponding deterministic model.

Case 2. We demonstrate the situation where the cholera infection in the deterministic model will persist while the infection in the stochastic model will be suppressed by noises when $\mathcal{R}_s < 1$ but $R_0 > 1$. Taking $\beta = 0.15$, $\tau_1 = 1.5$, $\tau_2 = 0.5$, $\tau_3 = 1.3$, and using the values of other parameters in Table 1, we get $R_0 = 2.381$ and $\mathcal{R}_s = 0.9726$. The initial values for both deterministic and stochastic models are the same (7,000, 1,000, 1.5). Figure 2 shows the solutions for both deterministic and stochastic models, where the left column is for the deterministic model and the disease persists; the right column is for the stochastic model and the disease dies out.

Case 3. We demonstrate the situation where the cholera infection in the stochastic model will persist when $\mathcal{R}_s > 1$ and $R_0 > 1$, and there is a stationary distribution for the stochastic model. We take $\beta = 0.15$, then use the values of other parameters

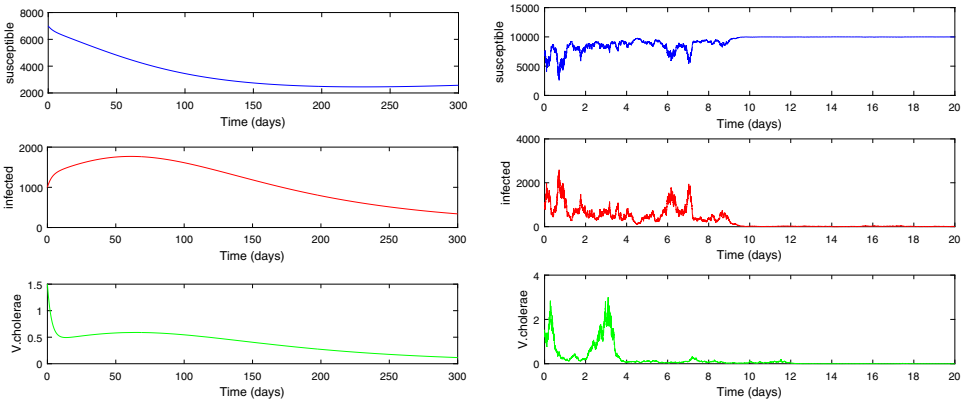

Fig. 2. Solution paths when $\mathcal{R}_s < 1$ but $R_0 > 1$. The left column is for the deterministic model where the infection persists; the right column is for the stochastic model where the infection dies out.

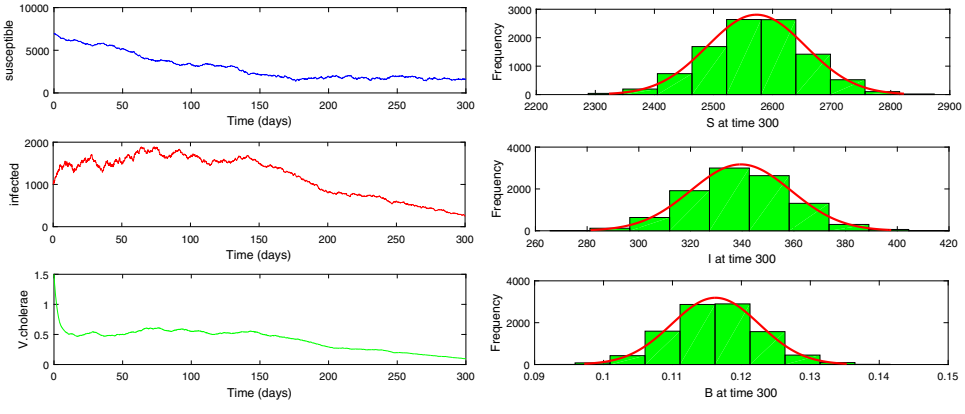

Fig. 3. Solution behaviors of the stochastic model when $\mathcal{R}_s > 1$ and $R_0 > 1$. The left column is one solution path; the right column are histograms for each population at time $t = 300$.

in Table 1, we get $R_0 = 2.381$ as in Case 2. Taking $\beta = 0.15$, $\tau_1 = \tau_2 = \tau_3 = 0.1$, and the values of other parameters in Table 1, we get $\mathcal{R}_s = 2.3726$ and the condition (3.2) of Theorem 3.1 is satisfied. The left column in Fig. 3 shows one path of the solutions for the stochastic model. At time $t = 300$, we simulate our stochastic model 10,000 times, and obtain frequency histograms for each population which show the stationary distributions. The frequency histograms are shown in the right column of Fig. 3.

5. Conclusions and Discussion

Cholera epidemics have been an important subject in public health, and mathematical modeling has provided some understanding for cholera spreading and controls.

In order to obtain detailed dynamics of cholera infections, the environmental fluctuations should be counted into modeling. Towards this direction, we take white-noise perturbations as a first approximate of environmental fluctuations. Based on a basic general deterministic mathematical model of cholera epidemics proposed in [23], we perturb nonlinear incidence function, the human birth/death rate, and the death rate of cholera vibrios in the aquatic resources with three mutually independent white noises, and then we have a stochastic model for cholera epidemics in terms of stochastic differential equations.

We identify almost sure positive invariant domain for our stochastic system, and show the solutions exist globally. We obtain a threshold $\mathcal{R}_s$ called the basic stochastic reproduction value which serves well as the basic reproduction number in the deterministic epidemic models. If the basic stochastic reproduction value is less than 1, then the solution tends asymptotically to the disease free equilibrium X_0 exponentially almost surely. That is, the cholera infections extinct almost surely. When the basic stochastic reproduction value is greater than 1, then the disease persists, and furthermore we show there is a unique stationary ergodic distribution to which all solutions eventually tend in distribution when the noises are bounded. It is also clear that the noises can suppress cholera infection spreading.

The basic stochastic reproduction value is the sum of the basic reproduction number (deterministic case) and a function of noise intensities and parameters, namely, $\mathcal{R}_s = R_0 - \frac{\delta}{m}$. To have a deep biological understanding, we recall that from the deterministic model to the stochastic model, $f(B)$, b, and m are replaced by $f(B) + \tau_1 \frac{I}{N} \frac{dW_1}{dt}$, $b + \tau_2 \frac{SI}{N^2} \frac{dW_2}{dt}$, and $m + \tau_3 \frac{dW_3}{dt}$, respectively. We may think that, each individual contributes environmental noise with intensity of τ_1 or τ_2. For the incidence function $f(B)$ which encodes interaction between human and cholera vibrios, it is reasonable to consider the contribution from infected individuals. So, we have $f(B) + \tau_1 \frac{I}{N} \frac{dW_1}{dt}$ in stochastic model. Similarly, for the human birth/death rate b, the environmental noise is mostly from infected and susceptible individuals, and for the convenience of analysis, we take the multiplication form $b + \tau_2 \frac{SI}{N^2} \frac{dW_2}{dt}$ in our stochastic model. For cholera vibrio death rate, the environment noise is encoded by $m + \tau_3 \frac{dW_3}{dt}$ since we consider the environment affect them as a whole. Somehow, the overall effect $\delta = \delta(\tau_1, \tau_2, \tau_3)$ needs to be divided by the death rate of cholera vibios m. Although the explicit form for $\delta(\tau_1, \tau_2, \tau_3)$ is unavailable, we may still deduce some information from its definition. When any one of three noise intensities increases, the basic stochastic reproduction value will decrease. The environmental noises reduce the threshold for the model dynamical patterns. Although we think each different noise should affect the model differently, because of implicit form $\delta(\tau_1, \tau_2, \tau_3)$, we cannot tell how each noise affects the model behaviors differently.

It is clear that there are other ways to include environmental fluctuations into modeling in general. For example, we may simply consider the environmental noise modeled by $\tau \frac{dW}{dt}$. However, in order to have a more complete dynamical picture of cholera epidemics, we should perturb other types of models including more realistic

incidence functions as proposed in [23], and to verify our thoughts about different noises would affect model dynamics differently.

As we mentioned, this study is our first step toward more realistic models for cholera epidemics which incorporate environmental fluctuations. There are several important aspects we need to study further in the future. First, we will perturb more parameters within any given model. Second, we will consider more general models in cholera epidemics. Third, we will consider how the perturbations affect the dynamical behaviors with different noises in general modeling setting.

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