ASYMPTOTIC BEHAVIOR OF THE STOCHASTIC
KELLER-SEGEL EQUATIONS

Yadong Shang

School of Mathematics and Information Sciences, Guangzhou University
Guangzhou 510006, China

Jianjun Paul Tian

Department of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88001, USA

Bixiang Wang*

Department of Mathematics, New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

Abstract. This paper deals with the asymptotic behavior of the solutions of
the non-autonomous one-dimensional stochastic Keller-Segel equations defined
in a bounded interval with Neumann boundary conditions. We prove the ex-
istence and uniqueness of tempered pullback random attractors under certain
conditions. We also establish the convergence of the solutions as well as the
pullback random attractors of the stochastic equations as the intensity of noise
approaches zero.

1. **Introduction.** In this paper, we investigate the long term dynamics of the non-
autonomous stochastic Keller-Segel equations defined in a bounded interval I for
$t > \tau$ with $\tau \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left(u \frac{\partial}{\partial x} f(\rho) \right), \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} = b \frac{\partial^2 \rho}{\partial x^2} + c(t)u - d\rho + \lambda \rho \circ \frac{dW}{dt}, \tag{1.2}$$

which are supplemented with homogeneous Neumann boundary conditions and
appropriate initial conditions. The unknown functions in system (1.1)-(1.2) are
$u = u(x,t)$ and $\rho = \rho(x,t)$, a, b and d are fixed positive constants, $c : \mathbb{R} \to \mathbb{R}^+$ is
a given function, $f : \mathbb{R} \to \mathbb{R}$ is a given nonlinearity. W is a two-sided real-valued
Wiener process defined on a probability space and $\lambda > 0$ is the intensity of noise.
The symbol $\circ$ in (2.2) indicates that the equation is understood in the sense of
Stratonovich's integration.

The deterministic version (i.e., $\lambda = 0$) of system (1.1)-(1.2) was proposed by
Keller and Segel in [26] to model the aggregation process of cellular slime mold by

chemical attraction. From biological point of view, u and ρ represent the population density of biological individuals and the concentration of chemical substance, respectively, a is the diffusion rate of u, b is the diffusion rate of ρ, c and d are the degradation and production rates of ρ, respectively. The nonlinear function f in (1.1) is called a sensitivity function that is used to model the response of of cells to chemicals. The term $-\frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}f(\rho)\right)$ is called a chemotactic term that is used to model the fact that cells are attracted by chemical stimulus. Several interesting nonlinear functions f are extensively investigated in the literature (see, e.g., [28, 32, 34]) including

$$f(s) = s, \quad s^2, \quad \ln(1+s), \quad \frac{s}{1+s}, \quad \text{and} \quad \frac{s^2}{1+s^2} \tag{1.3}$$

for $s \geq 0$.

The deterministic Keller-Segel equations have been studied by many experts, see, e.g., [16, 23, 24, 28, 30, 31, 32, 33, 34, 41]. In particular, the existence of solutions of the equations were investigated in [16, 30, 41], the blow-up of solutions were examined in [23, 24, 31], and the global attractors were discussed in [32, 33]. However, as far as the authors are aware, there is no result available in the literature regarding the long term dynamics of the stochastic Keller-Segel system given by (1.1)-(1.2). The goal of the present paper is to investigate this problem and establish the existence of tempered pullback random attractors for the stochastic system in an invariant subset of $L^2(I) \times H^1(I)$. We will also examine the limiting behavior of the solutions of system (1.1)-(1.2) as $\lambda \to 0$, and prove the convergence of the solutions as well as the pullback random attractors as $\lambda \to 0$. The main difficulty of the paper lies in how to derive pullback uniform estimates for the solutions. Since system (1.1)-(1.2) is a quasilinear system for the unknown functions u and ρ, it is hard to derive such estimates. We will combine the semigroup method and the energy method to establish the desired a priori uniform estimates for the stochastic system.

The concept of random attractors was introduced in [13, 15, 35] and further studied in [2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 17, 18, 19, 20, 25, 27, 29, 36, 37] for the autonomous stochastic equations; and in [1, 10, 14, 21, 22, 38, 39, 40] for the non-autonomous stochastic equations. We here will investigate the pullback random attractors for the non-autonomous stochastic system (1.1)-(1.2).

Notice that (1.1) is a deterministic equation which is not perturbed by noise. When system (1.1)-(1.2) is supplemented with homogeneous Neumann boundary conditions, by (1.1) we find that

$$\int_I u(x,t)dx \text{ is constant for all } t \geq \tau, \tag{1.4}$$

where τ is the initial time. This means that the total population of biological individuals is conserved for all $t \geq \tau$, a fact of significance in both biology and mathematics. If (1.1) is perturbed by white noise, then the solutions of the system do not satisfy (1.4) anymore, which is not consistent with the deterministic system from biological point of view, and also introduces difficulty to derive uniform estimates of the solutions. That is why we do not perturb (1.1) by white noise in this paper.

This paper is organized as follows. In the next section, we define a continuous cocycle for the non-autonomous stochastic system (1.1)-(1.2) in an invariant subset

of $L^2(I) \times H^1(I)$. In Section 3, we derive pullback uniform estimates for the solutions which are needed for constructing pullback random absorbing sets. We then prove the existence of pullback random attractors in Section 4, and establish the convergence of the solutions as well as the pullback random attractors as $\lambda \to 0$ in Section 5.

Hereafter, we use C and C_i $(i = 1, 2, \cdots,)$ to denote generic positive constants whose values may change from line to line.

For later purpose, we recall the following Gagliardo-Nirenberg interpolation inequality:

Lemma 1.1. *Let I be a bounded interval in $\mathbb{R}$. Suppose $s > 0$, $1 \leq q, r \leq \infty$, $m \in \{1, 2, \cdots\}$ and $j \in \{0, 1, 2, \cdots\}$. If*

$$\frac{j}{m} \leq \theta \leq 1$$

and

$$\frac{1}{p} = j + \theta(\frac{1}{r} - m) + \frac{1 - \theta}{q}$$

then there exists a positive constant $C = C(m, j, q, r, \theta, s, I)$ such that

$$\|D^j u\|_{L^p(I)} \leq C\|D^m u\|_{L^r(I)}^{\theta}\|u\|_{L^q(I)}^{1-\theta} + C\|u\|_{L^s(I)} \tag{1.5}$$

for all $u : I \to \mathbb{R}$ provided the right-hand side of (1.5) is finite.

Note that the space $H^s(I)$ is continuously embedded into $C(\bar{I})$ for $s > \frac{1}{2}$, that is, there exists a positive constant $C = C(s, I)$ such that

$$\|u\|_{C(\bar{I})} \leq C\|u\|_{H^s(I)}, \quad \forall\, u \in H^s(I). \tag{1.6}$$

The following Agmon's inequality will also be used in this paper:

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{L^2(I)}^{\frac{1}{2}}\|u\|_{H^1(I)}^{\frac{1}{2}}, \quad \forall\, u \in H^1(I), \tag{1.7}$$

for some $C > 0$.

2. **Cocycles for the stochastic Keller-Segel system.** In this section, we prove the global existence of solutions for the non-autonomous stochastic Keller-Segel system under certain conditions, and define a continuous cocycle in an invariant subset of $L^2(I) \times H^1(I)$.

Given $\tau \in \mathbb{R}$, consider the following one-dimensional stochastic Keller-Segel equations defined in a bounded interval $I = (a_1, b_1)$ for $t > \tau$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}f(\rho)\right), \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + c(t)u - d\rho + \lambda\rho \circ \frac{dW}{dt}, \tag{2.2}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(a_1, t) = \frac{\partial u}{\partial x}(b_1, t) = \frac{\partial \rho}{\partial x}(a_1, t) = \frac{\partial \rho}{\partial x}(b_1, t) = 0, \tag{2.3}$$

and initial conditions

$$u(x, \tau) = u_0(x), \quad \rho(x, \tau) = \rho_0(x), \tag{2.4}$$

where a, b, d and λ are all positive constants.

Throughout this paper, we will assume that $f : [0, \infty) \to \mathbb{R}$ is a smooth function such that there exist constants $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha > 0$ such that for all $s \geq 0$,

$$|f'(s)| + |f''(s)| + |f'''(s)| \leq \alpha_1 + \alpha_2 s^\alpha. \tag{2.5}$$

Note that all functions given by (1.3) satisfy condition (2.5).

We will also assume

$$c : \ \mathbb{R} \to \mathbb{R}^+ \quad \text{is continuous and bounded,} \tag{2.6}$$

where c is the function in (2.2).

To describe the Wiener process W, we introduce the standard Wiener space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \ \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Then the Wiener process W on $(\Omega, \mathcal{F}, P)$ takes the form: $W(t, \omega) = \omega(t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$. Denote by $\theta_t : \Omega \to \Omega$ the transformation

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$

Then by [2], $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system, and there exists a θ_t-invariant set of full measure (which is still denoted by Ω) such that for every ω in that set,

$$\lim_{t \to \pm\infty} \frac{\omega(t)}{t} = 0.$$

Next, we establish the existence and uniqueness of solutions of system (2.1)-(2.4) under (2.5). To that end, we need to transform the stochastic equation (2.2) into a deterministic one parametrized by the sample paths. Let $v(x, t) = e^{-\lambda \omega(t)} \rho(x, t)$. Then by (2.1)-(2.2) we find that u and v satisfy

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left(u \frac{\partial}{\partial x} f(e^{\lambda \omega(t)} v) \right), \quad t > \tau, \tag{2.7}$$

$$\frac{\partial v}{\partial t} = b \frac{\partial^2 v}{\partial x^2} - dv + c(t) e^{-\lambda \omega(t)} u, \quad t > \tau, \tag{2.8}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(a_1, t) = \frac{\partial u}{\partial x}(b_1, t) = \frac{\partial v}{\partial x}(a_1, t) = \frac{\partial v}{\partial x}(b_1, t) = 0, \quad t > \tau, \tag{2.9}$$

and initial conditions

$$u(x, \tau) = u_0(x), \quad v(x, \tau) = v_0(x), \tag{2.10}$$

with $v_0(x) = e^{-\lambda \omega(\tau)} \rho_0(x)$. By the Galerkin method, one can verify that if f satisfies (2.5), then problem (2.7)-(2.10) has a unique local solution for every $(u_0, v_0) \in L^2(I) \times H^1(I)$. More precisely, we have the following lemma.

Lemma 2.1. *Suppose* (2.5) *holds true. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(u_0, v_0) \in H^1(I)$ *with* $\|u_0\|_{L^2(I)} + \|v_0\|_{H^1} \leq R$ *for some* $R > 0$. *Then there exists a positive number* $T_0 = T_0(\tau, \omega, R)$ *such that problem* (2.7)-(2.10) *has a unique solution* $(u, v) = (u(t, \tau, \omega, u_0), v(t, \tau, \omega, v_0))$ *defined for* $t \in [\tau, \tau + T_0]$ *with the properties*

$$u \in C([\tau, \tau + T_0], L^2(I)) \bigcap L^2((\tau, \tau + T_0), H^1(I)), \quad \frac{du}{dt} \in L^2((\tau, \tau + T_0), H^{-1}(I))$$

and

$$v \in C([\tau, \tau + T_0], H^1(I)) \bigcap L^2((\tau, \tau + T_0), H^2(I)) \quad \text{and} \quad \frac{dv}{dt} \in L^2((\tau, \tau + T_0), L^2(I)).$$

In addition, $(u(t), v(t))$ is continuous with respect to initial data (u_0, v_0) in $\in L^2(I) \times H^1(I)$ and is measurable with respect to $\omega \in \Omega$ for every $t \in [\tau, \tau + T_0]$. Furthermore, if $u_0 \geq 0$ and $v_0 \geq 0$, then for every $t \in [\tau, \tau + T_0]$, $u(t) \geq 0$ and $v(t) \geq 0$.

Proof. The existence of local solutions follows from a standard process by applying the Galerkin method, see, e.g., [32]. The uniqueness and nonnegativity of solutions with nonnegative initial data can be obtained by the arguments of [32]. Since the local solution of problem (2.7)-(2.10) is given by the limit of the measurable solutions in ω of a family of finite-dimensional Galerkin systems, we infer that this local solution of system (2.7)-(2.10) is also measurable in $\omega \in \Omega$. □

In what follows, we prove the local solution of problem (2.7)-(2.10) obtained in Lemma 2.1 is actually defined for all $t \geq \tau$ when initial data are nonnegative. For that purpose, we only need to derive uniform estimates of the solutions on a finite time interval $[\tau, \tau + T]$ where the solution is defined. First, by integrating equation (2.7) we get

$$\frac{d}{dt} \int_I u(x, t) dx = 0,$$

which together with the nonnegativity of solutions implies

$$\|u(t)\|_{L^1(I)} = \|u_0\|_{L^1(I)}, \quad \forall\ t \in [\tau, \tau + T]. \tag{2.11}$$

Based on (2.11), we now derive uniform estimates on the component v of the solution (u, v) in $H^1(I)$.

Lemma 2.2. *Suppose (2.5) holds true. Let $\lambda_0 > 0$, $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_0, v_0) \in L^2(I) \times H^1(I)$ with $u_0 \geq 0$, $v_0 \geq 0$, and $\|u_0\|_{L^2(I)} + \|v_0\|_{H^1} \leq R$ for some $R > 0$. Then there exists a positive number $M_1 = M_1(\tau, T, \omega, R, \lambda_0)$ such that the solution (u, v) of problem (2.7)-(2.10) satisfies, for all $0 < \lambda \leq \lambda_0$,*

$$\|v(t, \tau, \omega, v_0)\|_{H^1(I)} \leq M_1 \quad for\ all\ \ t \in [\tau, \tau + T].$$

Proof. For convenience, we write $A = -b\partial_{xx} + d$ with domain $D(A) = \{v \in H^2(I) : v$ satisfies (2.9)$\}$. Given $\theta \geq 0$, let A^θ be the fractional power of A. It follows from [32] that $D(A^\theta) \subseteq H^{2\theta}(I)$ for $\theta \geq 0$, which along with (1.6) implies that for every $v_1 \in L^1(I)$ and $v_2 \in D(A^\theta)$ with $\theta > \frac{1}{4}$,

$$|\int_I v_1(x) v_2(x) dx| \leq \|v_2\|_{C(\bar{I})} \|v_1\|_{L^1(I)} \leq C\|v_2\|_{H^{2\theta}} \|v_1\|_{L^1(I)} \leq C\|v_2\|_{D(A^\theta)} \|v_1\|_{L^1(I)}. \tag{2.12}$$

By (2.12) we find that if $\theta > \frac{1}{4}$ and $v_1 \in L^1(I)$, then $v_1 \in D(A^{-\theta})$ and

$$\|v_1\|_{D(A^{-\theta})} \leq C\|v_1\|_{L^1(I)}, \tag{2.13}$$

where $D(A^{-\theta})$ is the dual space of $D(A^\theta)$.

Note that

$$\|e^{-At} v_0\|_{L^2(I)} \leq e^{-dt} \|v_0\|_{L^2(I)} \quad for\ all\ \ v_0 \in L^2(I)\ \ and\ t \geq 0. \tag{2.14}$$

Note that equation (2.8) can be reformulated as

$$\frac{dv}{dt} + Av = c(t) e^{-\lambda\omega(t)} u, \quad v(\tau) = v_0,$$

and hence

$$v(t) = e^{A(\tau - t)} v_0 + \int_\tau^t c(s) e^{A(s-t)} e^{-\lambda\omega(s)} u(s) ds. \tag{2.15}$$

By (2.13)-(2.15) we get, for $t \in [\tau, \tau + T]$,

$$\|A^{\frac{1}{2}}v(t)\|_{L^2(I)}$$

$$\leq e^{-d(t-\tau)}\|A^{\frac{1}{2}}v_0\|_{L^2(I)} + \int_\tau^t c(s)e^{-\lambda\omega(s)}\|e^{A(s-t)}A^{\frac{7}{8}}(A^{-\frac{3}{8}}u(s))\|_{L^2(I)}ds$$

$$\leq C_1 e^{-d(t-\tau)}\|v_0\|_{H^1(I)} + C_2 \int_\tau^t (t-s)^{-\frac{7}{8}}e^{-d(t-s)-\lambda\omega(s)}\|u(s)\|_{D(A^{-\frac{3}{8}})}ds$$

$$\leq C_1 e^{-d(t-\tau)}\|v_0\|_{H^1(I)} + C_3 \int_\tau^t (t-s)^{-\frac{7}{8}}e^{-d(t-s)-\lambda\omega(s)}\|u(s)\|_{L^1(I)}ds. \qquad (2.16)$$

By (2.11) and (2.16) we obtain, for $t \in [\tau, \tau + T]$,

$$\|A^{\frac{1}{2}}v(t)\|_{L^2(I)} \leq C_1 e^{-d(t-\tau)}\|v_0\|_{H^1(I)} + C_3\|u_0\|_{L^1(I)} \int_\tau^t (t-s)^{-\frac{7}{8}}e^{-d(t-s)-\lambda\omega(s)}ds$$

$$\tag{2.17}$$

$$\leq C_1 e^{-d(t-\tau)}\|v_0\|_{H^1(I)} + C_4\|u_0\|_{L^2(I)} \int_\tau^t (t-s)^{-\frac{7}{8}}e^{-d(t-s)-\lambda\omega(s)}ds,$$

from which the desired estimates follows. $\qquad \square$

Next, we derive uniform estimates on the component u of the solution (u, v) in $L^2(I)$.

Lemma 2.3. *Suppose* (2.5) *holds true. Let* $\lambda_0 > 0$, $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(u_0, v_0) \in L^2(I) \times H^1(I)$ *with* $u_0 \geq 0$, $v_0 \geq 0$, *and* $\|u_0\|_{L^2(I)} + \|v_0\|_{H^1} \leq R$ *for some* $R > 0$. *Then there exists a positive number* $M_2 = M_2(\tau, T, \omega, R, \lambda_0)$ *such that the solution* (u, v) *of problem* (2.7)-(2.10) *satisfies, for all* $0 < \lambda \leq \lambda_0$,

$$\|u(t, \tau, \omega, u_0)\|_{L^2(I)}^2 + \int_\tau^{\tau+T} (\|u(t)\|_{H^1(I)}^2 + \|v(t)\|_{H^2(I)}^2)dt \leq M_2 \quad for\ all\ \ t \in [\tau, \tau+T].$$

$$\tag{2.18}$$

Proof. By (2.7) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(I)}^2 + a\|u_x\|_{L^2(I)}^2 = \int_I uu_x \frac{\partial}{\partial x}\left(f(e^{\lambda\omega(t)}v)\right)dx$$

$$\leq \frac{1}{2}a\|u_x\|_{L^2(I)}^2 + \frac{1}{2a}e^{2\lambda\omega(t)} \int_I u^2|v_x|^2|f'(e^{\lambda\omega(t)}v)|^2 dx. \qquad (2.19)$$

We now estimate the last term on the right-hand side of (2.19). By (2.5) we get

$$\frac{1}{2a}e^{2\lambda\omega(t)} \int_I u^2|v_x|^2|f'(e^{\lambda\omega(t)}v)|^2 dx$$

$$\leq \frac{1}{2a}e^{2\lambda\omega(t)} \int_I (\alpha_1 + \alpha_2 e^{\alpha\lambda\omega(t)}|v|^\alpha)^2 u^2|v_x|^2 dx$$

$$\leq C_1 e^{2\lambda\omega(t)} \int_I u^2|v_x|^2 dx + C_2 e^{2(\alpha+1)\lambda\omega(t)} \int_I |v|^{2\alpha}u^2|v_x|^2 dx. \qquad (2.20)$$

To estimate the right-hand side of (2.20), we use the following interpolation inequalities from (1.5):

$$\|u\|_{L^4(I)} \leq C_3\|u\|_{H^1(I)}^{\frac{1}{2}}\|u\|_{L^1(I)}^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{L^4(I)} \leq C_4\|u\|_{H^1(I)}^{\frac{1}{4}}\|u\|_{L^2(I)}^{\frac{3}{4}}. \qquad (2.21)$$

By (1.6) and (2.21), for the second term on the right-hand side of (2.20) we have

$$C_2 e^{2(\alpha+1)\lambda\omega(t)} \int_I |v|^{2\alpha}u^2|v_x|^2 dx \leq C_2 e^{2(\alpha+1)\lambda\omega(t)}\|v\|_{L^\infty(I)}^{2\alpha} \int_I u^2|v_x|^2 dx$$

$$\leq C_5 e^{2(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{2\alpha}\|u\|_{L^4(I)}^2\|v_x\|_{L^4(I)}^2$$

$$\leq C_6 e^{2(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{2\alpha}\|u\|_{H^1(I)}\|u\|_{L^1(I)}\|v_x\|_{L^2(I)}^{\frac{3}{2}}\|v_x\|_{H^1(I)}^{\frac{1}{2}}$$

$$\leq C_7 e^{2(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{2\alpha+2}\|u\|_{H^1(I)}\|u\|_{L^1(I)}$$

$$+ C_7 e^{2(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{2\alpha+\frac{3}{2}}\|u\|_{H^1(I)}\|u\|_{L^1(I)}\|v_{xx}\|_{L^2(I)}^{\frac{1}{2}}. \tag{2.22}$$

For convenience, we write

$$\delta = \min\{\frac{1}{2}a, b, d\}. \tag{2.23}$$

Then by Young's inequality, (2.11) and (2.22) we get

$$C_2 e^{2(\alpha+1)\lambda\omega(t)}\int_I |v|^{2\alpha}u^2|v_x|^2 dx$$

$$\leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + C_8 e^{4(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{4\alpha+4} + C_8 e^{4(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{4\alpha+3}\|v_{xx}\|_{L^2(I)}$$

$$\leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + \frac{1}{8}\delta\|v_{xx}\|_{L^2(I)}^2 + C_8 e^{4(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{4\alpha+4} + C_9 e^{8(\alpha+1)\lambda\omega(t)}\|v\|_{H^1(I)}^{8\alpha+6}$$

$$\leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + \frac{1}{8}\delta\|v_{xx}\|_{L^2(I)}^2 + C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}. \tag{2.24}$$

By the process to derive (2.24), we also obtain

$$C_1 e^{2\lambda\omega(t)}\int_I u^2|v_x|^2 dx$$

$$\leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + \frac{1}{8}\delta\|v_{xx}\|_{L^2(I)}^2 + C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + C_{12}e^{12\lambda\omega(t)},$$

which along with (2.20) and (2.24) implies

$$\frac{1}{2a}e^{2\lambda\omega(t)}\int_I u^2|v_x|^2|f'(e^{\lambda\omega(t)}v)|dx$$

$$\leq \frac{1}{4}\delta\|u\|_{H^1(I)}^2 + \frac{1}{4}\delta\|v_{xx}\|_{L^2(I)}^2 + C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+ C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + C_{12}e^{12\lambda\omega(t)} \tag{2.25}$$

By (2.19) and (2) we obtain

$$\frac{d}{dt}\|u\|_{L^2(I)}^2 + a\|u_x\|_{L^2(I)}^2 \leq \frac{1}{2}\delta\|u\|_{H^1(I)}^2 + \frac{1}{2}\delta\|v_{xx}\|_{L^2(I)}^2$$

$$+ 2C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+ 2C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + 2C_{12}e^{12\lambda\omega(t)}. \tag{2.26}$$

Note that by (1.5),

$$\|u\|_{L^2(I)} \leq C_{13}\|u\|_{H^1(I)}^{\frac{1}{3}}\|u\|_{L^1(I)}^{\frac{2}{3}}. \tag{2.27}$$

By (2.27) we get

$$2d\|u\|_{L^2(I)}^2 \leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + C_{14}\|u\|_{L^1(I)}^2,$$

which along with (2.11) and (2.26) yields

$$\frac{d}{dt}\|u\|_{L^2(I)}^2 + 2d\|u\|_{L^2(I)}^2 + a\|u_x\|_{L^2(I)}^2 \leq \frac{5}{8}\delta\|u\|_{H^1(I)}^2 + \frac{1}{2}\delta\|v_{xx}\|_{L^2(I)}^2$$

$$+ 2C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+2C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + 2C_{12}e^{12\lambda\omega(t)} + C_{15}. \tag{2.28}$$

On the other hand, by (2.8) we get

$$\frac{1}{2}\frac{d}{dt}\|v_x\|_{L^2(I)}^2 + b\|v_{xx}\|_{L^2(I)}^2 + d\|v_x\|_{L^2(I)}^2 = -e^{-\lambda\omega(t)}\int_I c(t)uv_{xx}dx$$

$$\leq \frac{1}{2}b\|v_{xx}\|_{L^2(I)}^2 + C_{16}e^{-2\lambda\omega(t)}\|u\|_{L^2(I)}^2. \tag{2.29}$$

By (2.27) we obtain

$$C_{16}e^{-2\lambda\omega(t)}\|u\|_{L^2(I)}^2 \leq \frac{1}{8}\delta\|u\|_{H^1(I)}^2 + C_{17}e^{-3\lambda\omega(t)}\|u\|_{L^1(I)}^2. \tag{2.30}$$

By (2.11) and (2.29)-(2.30) we get

$$\frac{d}{dt}\|v_x\|_{L^2(I)}^2 + b\|v_{xx}\|_{L^2(I)}^2 + 2d\|v_x\|_{L^2(I)}^2 \leq \frac{1}{4}\delta\|u\|_{H^1(I)}^2 + C_{18}e^{-3\lambda\omega(t)}. \tag{2.31}$$

It follows from (2) and (2.31) that

$$\frac{d}{dt}(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + 2d(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + a\|u_x\|_{L^2(I)}^2 + b\|v_{xx}\|_{L^2(I)}^2$$

$$\leq \frac{7}{8}\delta\|u\|_{H^1(I)}^2 + \frac{1}{2}\delta\|v_{xx}\|_{L^2(I)}^2$$

$$+2C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+2C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + 2C_{12}e^{12\lambda\omega(t)} + C_{18}e^{-3\lambda\omega(t)} + C_{15}. \tag{2.32}$$

By (2.23) and (2) we find that

$$\frac{d}{dt}(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + d(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + \frac{1}{2}a\|u_x\|_{L^2(I)}^2 + \frac{1}{2}b\|v_{xx}\|_{L^2(I)}^2$$

$$\leq 2C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+2C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + 2C_{12}e^{12\lambda\omega(t)} + C_{18}e^{-3\lambda\omega(t)} + C_{15}. \tag{2.33}$$

By (2) and Lemma 2.2 we infer that there exists $C_{19} = C_{19}(\tau, T, \omega, R, \lambda_0) > 0$ such that for all $t \in [\tau, \tau + T]$ and $0 < \lambda \leq \lambda_0$,

$$\frac{d}{dt}(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + \frac{1}{2}\min\{a, b\}(\|u_x\|_{L^2(I)}^2 + \|v_{xx}\|_{L^2(I)}^2) \leq C_{19},$$

from which (2.18) follows.   □

As an immediate consequence of Lemmas 2.1, 2.2 and 2.3 we obtain the global existence of solutions for problem (2.7)-(2.10).

Corollary 2.4. *Suppose* (2.5) *holds true. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(u_0, v_0) \in L^2(I) \times H^1(I)$ *with* $u_0 \geq 0$ *and* $v_0 \geq 0$. *Then system* (2.7)-(2.10) *possesses a unique nonnegative solution* $(u, v) = (u(t, \tau, \omega, u_0), v(t, \tau, \omega, v_0))$ *defined for all* $t \geq \tau$ *with the properties*

$$u \in C([\tau, \infty), L^2(I)) \bigcap L^2_{loc}((\tau, \infty), H^1(I)), \quad \frac{du}{dt} \in L^2_{loc}((\tau, \infty), H^{-1}(I))$$

and

$$v \in C([\tau, \infty), H^1(I)) \bigcap L^2_{loc}((\tau, \infty), H^2(I)) \quad and \quad \frac{dv}{dt} \in L^2_{loc}((\tau, \infty), L^2(I)).$$

In addition, $(u(t), v(t))$ *is continuous with respect to initial data* (u_0, v_0) *in* $\in L^2(I) \times H^1(I)$ *and is measurable with respect to* $\omega \in \Omega$ *for every* $t \geq \tau$.

Next, we establish the uniform estimates of solutions of problem (2.7)-(2.10) in $H^1(I) \times H^2(I)$.

Lemma 2.5. *Suppose* (2.5) *holds true. Let* $\lambda_0 > 0$, $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(u_0, v_0) \in L^2(I) \times H^1(I)$ *with* $u_0 \geq 0$, $v_0 \geq 0$, *and* $\|u_0\|_{L^2(I)} + \|v_0\|_{H^1} \leq R$ *for some* $R > 0$. *Then there exists a positive number* $M_3 = M_3(\tau, T, \omega, R, \lambda_0)$ *such that the solution* (u, v) *of problem* (2.7)-(2.10) *satisfies, for all* $0 < \lambda \leq \lambda_0$,

$$\|u(t, \tau, \omega, u_0)\|_{H^1(I)}^2 + \|v(t, \tau, \omega, u_0)\|_{H^2(I)}^2 \leq M_3 + M_3(t - \tau)^{-1} \quad \text{for all } t \in (\tau, \tau + T].$$
(2.34)

Proof. By (2.8) we get

$$\frac{1}{2}\frac{d}{dt}\|v_{xx}\|_{L^2(I)}^2 + b\|v_{xxx}\|_{L^2(I)}^2 + d\|v_{xx}\|_{L^2(I)}^2 = -c(t)e^{-\lambda\omega(t)}\int_I u_x v_{xxx} dx$$

$$\leq \frac{1}{2}b\|v_{xxx}\|_{L^2(I)}^2 + \frac{1}{2b}c^2(t)e^{-2\lambda\omega(t)}\|u_x\|_{L^2(I)}^2$$

which implies

$$\frac{d}{dt}\|v_{xx}\|_{L^2(I)}^2 + b\|v_{xxx}\|_{L^2(I)}^2 + 2d\|v_{xx}\|_{L^2(I)}^2 \leq \frac{1}{b}c^2(t)e^{-2\lambda\omega(t)}\|u_x\|_{L^2(I)}^2.$$
(2.35)

By (2.7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2(I)}^2 + a\|u_{xx}\|_{L^2(I)}^2 = \int_I u_{xx}\frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v)\right)$$

$$\leq \frac{1}{2}a\|u_{xx}\|_{L^2(I)}^2 + \frac{1}{2a}\int_I \left(u_x\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v) + u\frac{\partial^2}{\partial x^2}f(e^{\lambda\omega(t)}v)\right)^2 dx$$

which implies

$$\frac{d}{dt}\|u_x\|_{L^2(I)}^2 + a\|u_{xx}\|_{L^2(I)}^2$$

$$\leq \frac{3}{a}\int_I \left((e^{\lambda\omega(t)}f'(e^{\lambda\omega(t)}v)u_x v_x)^2 + (e^{\lambda\omega(t)}f'(e^{\lambda\omega(t)}v)uv_{xx})^2\right) dx$$

$$+ \frac{3}{a}\int_I (e^{2\lambda\omega(t)}f''(e^{\lambda\omega(t)}v)v_x^2 u)^2 dx.$$
(2.36)

We now estimate the right-hand side of (2). By (1.6) and Lemma 2.2 we find that there exists $C_1 = C_1(\tau, T, \omega, R, \lambda_0) > 0$ such that for all $0 < \lambda \leq \lambda_0$,

$$\|v(t)\|_{C(\bar{I})} \leq C_1 \quad \text{for all } t \in [\tau, \tau + T].$$
(2.37)

By (2.5) and (2.37) we obtain from (2) that, for all $0 < \lambda \leq \lambda_0$,

$$\frac{d}{dt}\|u_x\|_{L^2(I)}^2 + a\|u_{xx}\|_{L^2(I)}^2 \leq C_2 \int_I \left(u_x^2 v_x^2 + u^2 v_{xx}^2 + v_x^4 u^2\right) dx$$
(2.38)

for some $C_2 = C_2(\tau, T, \omega, R, \lambda_0) > 0$. By (1.5) we have

$$\|u_x\|_{L^4(I)} \leq C_3\|u_{xx}\|_{L^2(I)}^{\frac{5}{8}}\|u\|_{L^2(I)}^{\frac{3}{8}} + C_3\|u\|_{L^2(I)},$$
(2.39)

and

$$\|u\|_{L^4(I)} \leq C_4\|u_{xx}\|_{L^2(I)}^{\frac{1}{8}}\|u\|_{L^2(I)}^{\frac{7}{8}} + C_4\|u\|_{L^2(I)}.$$
(2.40)

By (2.39)-(2.40) and Lemmas 2.2 and 2.3 we obtain

$$C_2\int_I u_x^2 v_x^2 dx \leq C_2\|u_x\|_{L^4(I)}^2\|v_x\|_{L^4(I)}^2$$

$$\leq C_5(\|u_{xx}\|_{L^2(I)}^{\frac{5}{4}}\|u\|_{L^2(I)}^{\frac{3}{4}} + \|u\|_{L^2(I)}^2)(\|v_{xxx}\|_{L^2(I)}^{\frac{1}{4}}\|v_x\|_{L^2(I)}^{\frac{7}{4}} + \|v_x\|_{L^2(I)}^2)$$

$$\leq C_6(1+\|u_{xx}\|_{L^2(I)}^{\frac{5}{4}})(1+\|v_{xxx}\|_{L^2(I)}^{\frac{1}{4}}) \leq \frac{1}{4}a\|u_{xx}\|_{L^2(I)}^2+\frac{1}{4}b\|v_{xxx}\|_{L^2(I)}^2+C_7. \quad (2.41)$$

Similarly, by (2.39)-(2.40) we can also obtain

$$C_2\int_I u^2 v_{xx}^2 dx \leq C_8(1+\|u_{xx}\|_{L^2(I)}^{\frac{1}{4}})(1+\|v_{xxx}\|_{L^2(I)}^{\frac{5}{4}})$$

$$\leq \frac{1}{4}a\|u_{xx}\|_{L^2(I)}^2 + \frac{1}{4}b\|v_{xxx}\|_{L^2(I)}^2 + C_9, \quad (2.42)$$

and by (1.7) and (2.39)-(2.40),

$$C_2\int_I v_x^4 u^2 dx \leq C_{10}\|u\|_{L^\infty(I)}^2\|v_x\|_{L^4(I)}^4$$

$$\leq C_{11}(1+\|u_x\|_{L^2(I)})(1+\|v_{xxx}\|_{L^2(I)}^{\frac{1}{2}}) \leq \|u_x\|_{L^2(I)}^2 + \frac{1}{4}b\|v_{xxx}\|_{L^2(I)}^2 + C_{12}. \quad (2.43)$$

It follows from (2.38) and (2.41)-(2) that

$$\frac{d}{dt}\|u_x\|_{L^2(I)}^2 + \frac{1}{2}a\|u_{xx}\|_{L^2(I)}^2 \leq \frac{3}{4}b\|v_{xxx}\|_{L^2(I)}^2 + \|u_x\|_{L^2(I)}^2 + C_{13},$$

which along with (2.35) yields

$$\frac{d}{dt}(\|u_x\|_{L^2(I)}^2 + \|v_{xx}\|_{L^2(I)}^2) + \frac{1}{2}a\|u_{xx}\|_{L^2(I)}^2 + \frac{1}{4}b\|v_{xxx}\|_{L^2(I)}^2 + 2d\|v_{xx}\|_{L^2(I)}^2$$

$$\leq C_{14}\|u_x\|_{L^2(I)}^2 + C_{13}. \quad (2.44)$$

Let $t\in(\tau,\tau+T]$ and $s\in(\tau,t)$. Integrating (2.44) on (s,t) we get

$$\|u_x(t)\|_{L^2(I)}^2 + \|v_{xx}(t)\|_{L^2(I)}^2$$

$$\leq \|u_x(s)\|_{L^2(I)}^2 + \|v_{xx}(s)\|_{L^2(I)}^2 + C_{14}\int_\tau^{\tau+T}\|u_x(r)\|_{L^2(I)}^2 dr + C_{13}T.$$

We now integrate the above with respect to s on (τ,t) to obtain

$$(t-\tau)(\|u_x(t)\|_{L^2(I)}^2 + \|v_{xx}(t)\|_{L^2(I)}^2)$$

$$\leq \int_\tau^t (\|u_x(s)\|_{L^2(I)}^2 + \|v_{xx}(s)\|_{L^2(I)}^2)ds + C_{14}T\int_\tau^{\tau+T}\|u_x(r)\|_{L^2(I)}^2 dr + C_{13}T^2.$$

$$\leq \int_\tau^{\tau+T} (\|u_x(s)\|_{L^2(I)}^2 + \|v_{xx}(s)\|_{L^2(I)}^2)ds + C_{14}T\int_\tau^{\tau+T}\|u_x(r)\|_{L^2(I)}^2 dr + C_{13}T^2,$$

which along with Lemmas 2.2 and 2.3 yields the desired estimates. $\square$

As a consequence of Lemma 2.5 and the compactness of Sobolev embedding $H^1(I)\times H^2(I)\hookrightarrow L^2(I)\times H^1(I)$, we obtain the compactness of the solution operator of problem (2.7)-(2.10).

Corollary 2.6. *Suppose* (2.5) *holds true. Then given* $\tau\in\mathbb{R}$, $t>\tau$, $\omega\in\Omega$, *and a bounded sequence* $\{(u_{0,n},v_{0,n})\}_{n=1}^\infty$ *of nonnegative initial data in* $L^2(I)\times H^1(I)$, *the sequence* $\{(u(t,\tau,\omega,u_{0,n}),v(t,\tau,\omega,v_{0,n}))\}_{n=1}^\infty$ *of the solutions of problem* (2.7)-(2.10) *has a convergent subsequence in* $L^2(I)\times H^1(I)$.

Now by the solution (u, v) of (2.7)-(2.10), we can get a solution (u, ρ) for the stochastic system (2.1)-(2.4) where ρ is given by

$$\rho(t, \tau, \omega, \rho_0) = e^{\lambda \omega(t)} v(t, \tau, \omega, v_0) \tag{2.45}$$

with $\rho_0 = e^{\lambda \omega(\tau)} v_0$. By Corollary 2.4 we find that for every $(u_0, \rho_0) \in L^2(I) \times H^1(I)$ with $u_0 \geq 0$ and $\rho_0 \geq 0$, system (2.1)-(2.4) has a unique nonnegative solution $(u(t, \tau, \omega, u_0), \rho(t, \tau, \omega, \rho_0))$ in $L^2(I) \times H^1(I)$ which is defined for all $t \geq \tau$. This solution is both continuous in $t \in [\tau, \infty)$ and in $(u_0, \rho_0) \in L^2(I) \times H^1(I)$. Moreover, $(u(t, \tau, \cdot, u_0), \rho(t, \tau, \cdot, \rho_0)) : \Omega \to L^2(I) \times H^1(I)$ is measurable. Let γ be a fixed positive number, and define a subset of $L^2(I) \times H^1(I)$ by

$$H = \{(u, \rho) \in L^2(I) \times H^1(I) : u \geq 0, \ \rho \geq 0, \ \int_I u(x) dx \leq \gamma\}.$$

Then we see that H is invariant under the solution operator of system (2.1)-(2.4).

We now define a continuous cocycle in H for (2.1)-(2.4). Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ be a mapping given by, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_0, \rho_0) \in H$,

$$\Phi(t, \tau, \omega, (u_0, \rho_0)) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_0), \rho(t + \tau, \tau, \theta_{-\tau}\omega, \rho_0))$$

$$= (u(t + \tau, \tau, \theta_{-\tau}\omega, u_0), e^{\lambda(\omega(t) - \omega(-\tau))} v(t + \tau, \tau, \theta_{-\tau}\omega, v_0)) \tag{2.46}$$

where $v_0 = e^{\lambda \omega(-\tau)} \rho_0$. We will investigate the tempered random attractors for Φ in H.

Let $D = \{D(\tau, \omega) \subseteq H : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of H. Such a family D is called tempered if for every $C > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{Ct} \|D(\tau + t, \theta_t \omega)\| = 0,$$

where the norm $\|D\|$ of a set D in H is given by $\|D\| = \sup_{(u, \rho) \in D} \|(u, \rho)\|_{L^2(I) \times H^1(I)}$.

We will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of H:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq H : \tau \in \mathbb{R}, \omega \in \Omega\} : \ D \text{ is tempered in } H\}.$$

3. **Uniform estimates.** In this section, we derive uniform estimates for the cocycle Φ defined by (2). These estimates will be used to construct tempered pullback absorbing sets for system (2.1)-(2.4). We start with the uniform estimates on the component v of problem (2.7)-(2.10) in $H^1(I)$.

Lemma 3.1. *Suppose* (2.5) *holds true. Then for every* $\lambda_0 > 0$, $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D \in \mathcal{D}$, *there exists* $T_1 = T_1(\tau, \omega, D, \lambda_0, \sigma) > 0$ *such that for all* $t \geq T_1$ *and* $0 < \lambda \leq \lambda_0$, *the solution* (u, v) *of problem* (2.7)-(2.10) *satisfies*

$$\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_0)\|_{H^1(I)} \leq L_1 + L_1 \int_{-\infty}^{\sigma - \tau} (\sigma - \tau - s)^{-\frac{7}{8}} e^{d(s - \sigma + \tau)} e^{\lambda(\omega(-\tau) - \omega(s))} ds,$$

$$\tag{3.1}$$

where $(u_0, e^{\lambda(\omega(-t) - \omega(-\tau))} v_0) \in D(\tau - t, \theta_{-t}\omega)$, *and* L_1 *is a positive constant independent of* τ, ω, D *and* λ.

Proof. By replacing t by σ, τ by $\tau - t$ and ω by $\theta_{-\tau}\omega$ in (2.17) we obtain, for all $0 < \lambda \leq \lambda_0$,

$$\|A^{\frac{1}{2}} v(\sigma, \tau - t, \theta_{-\tau}\omega, v_0)\|_{L^2(I)} \leq C_1 e^{-d(\sigma - \tau + t)} \|v_0\|_{H^1(I)}$$

$$+ C_3 \|u_0\|_{L^1(I)} \int_{\tau - t}^{\sigma} (\sigma - s)^{-\frac{7}{8}} e^{-d(\sigma - s)} e^{\lambda(\omega(-\tau) - \omega(s - \tau))} ds$$

$$\leq C_1 e^{-d(\sigma-\tau)+\lambda\omega(-\tau)} e^{-dt-\lambda\omega(-t)} \|D(\tau-t,\theta_{-\tau}\omega)\|$$

$$+C_3\gamma \int_{-\infty}^{\sigma-\tau} (\sigma-\tau-s)^{-\frac{7}{8}} e^{d(s-\sigma+\tau)} e^{\lambda(\omega(-\tau)-\omega(s))} ds$$

$$\leq C_1 e^{-d(\sigma-\tau)+\lambda_0|\omega(-\tau)|} e^{-dt+\lambda_0|\omega(-t)|} \|D(\tau-t,\theta_{-\tau}\omega)\|$$

$$+C_3\gamma \int_{-\infty}^{\sigma-\tau} (\sigma-\tau-s)^{-\frac{7}{8}} e^{d(s-\sigma+\tau)} e^{\lambda(\omega(-\tau)-\omega(s))} ds. \tag{3.2}$$

Since $D \in \mathcal{D}$, we find that

$$\lim_{t\to\infty} C_1 e^{-d(\sigma-\tau)+\lambda_0|\omega(-\tau)|} e^{-dt+\lambda_0|\omega(-t)|} \|D(\tau-t,\theta_{-\tau}\omega)\| = 0,$$

and hence there exists $T_1 = T_1(\tau,\omega,D,\lambda_0,\sigma) > 0$ such that for all $t \geq T_1$ and $0 < \lambda \leq \lambda_0$,

$$C_1 e^{-d(\sigma-\tau)+\lambda_0|\omega(-\tau)|} e^{-dt+\lambda_0|\omega(-t)|} \|D(\tau-t,\theta_{-\tau}\omega)\| \leq 1. \tag{3.3}$$

Then (3.1) follows from (3.2)-(3.3). □

We now establish the uniform estimates on the component u of problem (2.7)-(2.10) in $L^2(I)$.

Lemma 3.2. *Suppose* (2.5) *holds true. Then for every* $\lambda_0 > 0$, $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D \in \mathcal{D}$, *there exists* $T_2 = T_2(\tau,\omega,D,\lambda_0,\sigma) > 0$ *such that for all* $t \geq T_2$ *and* $0 < \lambda \leq \lambda_0$, *the solution* (u,v) *of problem* (2.7)-(2.10) *satisfies*

$$\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_0)\|_{L^2(I)}^2 \leq L_2 + L_2 \int_{-\infty}^{\sigma} e^{d(r-\sigma)}\phi_\lambda(r,\theta_{-\tau}\omega)dr$$

$$+ L_2 \int_{-\infty}^{\sigma} e^{d(r-\sigma)} \left(\int_0^{\infty} s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-\tau-s))} ds \right)^{16\alpha+12} dr, \tag{3.4}$$

where $(u_0, e^{\lambda(\omega(-t)-\omega(-\tau))}v_0) \in D(\tau-t,\theta_{-t}\omega)$, L_2 *is a positive constant independent of* τ, ω, D *and* λ, *and*

$$\phi_\lambda(r,\omega) = e^{16(\alpha+1)\lambda\omega(r)} + e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(r)} + e^{\frac{\lambda(8\alpha+6)}{\alpha}\omega(r)} + e^{12\lambda\omega(r)} + e^{-3\lambda\omega(r)} + 1. \tag{3.5}$$

Proof. By (2) we find

$$\frac{d}{dt}(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + d(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2)$$

$$\leq 2C_9(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+2C_{11}(1 + e^{8\lambda\omega(t)})\|v\|_{H^1(I)}^6 + 2C_{12}e^{12\lambda\omega(t)} + C_{18}e^{-3\lambda\omega(t)} + C_{15}$$

$$\leq C_{19}(1 + e^{8(\alpha+1)\lambda\omega(t)})\|v\|_{H^1(I)}^{8\alpha+6} + 2C_{10}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+C_{20}(1 + e^{\frac{\lambda(8\alpha+6)}{\alpha}\omega(t)}) + 2C_{12}e^{12\lambda\omega(t)} + C_{18}e^{-3\lambda\omega(t)} + C_{15}.$$

$$\leq \|v\|_{H^1(I)}^{16\alpha+12} + C_{21}e^{16(\alpha+1)\lambda\omega(t)} + C_{21}e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}\omega(t)}$$

$$+ C_{21}e^{\frac{\lambda(8\alpha+6)}{\alpha}\omega(t)} + C_{21}e^{12\lambda\omega(t)} + C_{21}e^{-3\lambda\omega(t)} + C_{21}. \tag{3.6}$$

By (3.5) and (3.6) we get

$$\frac{d}{dt}(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) + d(\|u\|_{L^2(I)}^2 + \|v_x\|_{L^2(I)}^2) \leq \|v\|_{H^1(I)}^{16\alpha+12} + C_{21}\phi_\lambda(t,\omega). \tag{3.7}$$

Multiplying (3.7) by e^{dt}, and then integrating on $(\tau - t, \sigma)$ we get

$$\|u(\sigma, \tau - t, \omega, u_0)\|_{L^2(I)}^2 + \|v_x(\sigma, \tau - t, \omega, v_0)\|_{L^2(I)}^2$$

$$\leq e^{d(\tau - t - \sigma)}(\|u_0\|_{L^2(I)}^2 + \|v_0\|_{H^1(I)}^2) + \int_{\tau - t}^{\sigma} e^{d(r - \sigma)} \|v(r, \tau - t, \omega, v_0)\|_{H^1(I)}^{16\alpha + 12} dr$$

$$+ C_{21} \int_{\tau - t}^{\sigma} e^{d(r - \sigma)} \phi_\lambda(r, \omega) dr. \tag{3.8}$$

Replacing ω by $\theta_{-\tau}\omega$ in (3.8) we obtain that for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\tau - t \leq \sigma$,

$$\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_0)\|_{L^2(I)}^2 + \|v_x(\sigma, \tau - t, \theta_{-\tau}\omega, v_0)\|_{L^2(I)}^2$$

$$\leq e^{d(\tau - t - \sigma)}(\|u_0\|_{L^2(I)}^2 + \|v_0\|_{H^1(I)}^2) + \int_{\tau - t}^{\sigma} e^{d(r - \sigma)} \|v(r, \tau - t, \theta_{-\tau}\omega, v_0)\|_{H^1(I)}^{16\alpha + 12} dr$$

$$+ C_{21} \int_{\tau - t}^{\sigma} e^{d(r - \sigma)} \phi_\lambda(r, \theta_{-\tau}\omega) dr. \tag{3.9}$$

Since $(u_0, e^{\lambda(\omega(-t) - \omega(-\tau))} v_0) \in D(\tau - t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, for the first term on the right-hand side of (3.9) we have

$$e^{d(\tau - t - \sigma)}(\|u_0\|_{L^2(I)}^2 + \|v_0\|_{H^1(I)}^2) \leq e^{d(\tau - t - \sigma)}(1 + e^{2\lambda(\omega(-\tau) - \omega(-t))}) \|D(\tau - t, \theta_{-t}\omega)\|^2$$

$$\leq e^{d(\tau - t - \sigma)}(1 + e^{2\lambda_0(|\omega(-\tau)| + |\omega(-t)|)}) \|D(\tau - t, \theta_{-t}\omega)\|^2 \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, there exists $T_2 = T_2(\tau, \omega, D, \lambda_0, \sigma) > 0$ such that for all $t \geq T_2$ and $0 < \lambda \leq \lambda_0$,

$$e^{d(\tau - t - \sigma)}(\|u_0\|_{L^2(I)}^2 + \|v_0\|_{H^1(I)}^2) \leq 1. \tag{3.10}$$

Similarly, we have

$$\lim_{t \to \infty} e^{d\tau + \lambda_0|\omega(-\tau)|} e^{\lambda_0|\omega(-t)|} e^{-\frac{dt}{32\alpha + 24}} \|D(\tau - t, \theta_{-\tau}\omega)\| = 0,$$

and hence there exists $T_3 = T_3(\tau, \omega, D, \lambda_0) > 0$ such that for all $t \geq T_3$,

$$e^{d\tau + \lambda_0|\omega(-\tau)|} e^{\lambda_0|\omega(-t)|} e^{-\frac{dt}{32\alpha + 24}} \|D(\tau - t, \theta_{-\tau}\omega)\| \leq 1. \tag{3.11}$$

By (3.2) and (3.11) we get, for all $t \geq T_3$ and $0 < \lambda \leq \lambda_0$,

$$\|v(r, \tau - t, \theta_{-\tau}\omega, v_0)\|_{H^1(I)}$$

$$\leq C_{22} e^{-d(r+t)} e^{\frac{dt}{32\alpha + 24}} + C_{22} \int_{-\infty}^{r-\tau} (r - \tau - s)^{-\frac{7}{8}} e^{d(s-r+\tau)} e^{\lambda(\omega(-\tau) - \omega(s))} ds$$

$$\leq C_{22} e^{-d(r+t)} e^{\frac{dt}{32\alpha + 24}} + C_{22} \int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r - \tau - s))} ds. \tag{3.12}$$

Let $T_4 = \max\{T_2, T_3\}$. Then by (3.9), (3.10) and (3.12) we obtain, for all $t \geq T_4$ and $0 < \lambda \leq \lambda_0$,

$$\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_0)\|_{L^2(I)}^2 + \|v_x(\sigma, \tau - t, \theta_{-\tau}\omega, v_0)\|_{L^2(I)}^2$$

$$\leq 1 + C_{23} e^{-d\sigma} e^{\frac{1}{2}dt} e^{-dt(16\alpha + 12)} \int_{\tau - t}^{\sigma} e^{-dr(16\alpha + 11)} \mathrm{d}r$$

$$+ C_{23} e^{-d\sigma} \int_{\tau - t}^{\sigma} e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r - \tau - s))} ds \right)^{16\alpha + 12} \mathrm{d}r$$

$$+ C_{21} \int_{\tau - t}^{\sigma} e^{d(r - \sigma)} \phi_\lambda(r, \theta_{-\tau}\omega) \mathrm{d}r$$

$$\leq 1 + \frac{C_{23}}{d(16\alpha + 11)} e^{-d\sigma} e^{-d\tau(16\alpha+11)} e^{-\frac{1}{2}dt}$$

$$+ C_{23} e^{-d\sigma} \int_{\tau-t}^{\sigma} e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-\tau-s))} ds \right)^{16\alpha+12} dr$$

$$+ C_{21} \int_{\tau-t}^{\sigma} e^{d(r-\sigma)} \phi_\lambda(r, \theta_{-\tau}\omega) dr. \tag{3.13}$$

Note that there exists $T_5 = T_5(\tau, \sigma) > 0$ such that for all $t \geq T_5$,

$$\frac{C_{23}}{d(16\alpha + 11)} e^{-d\sigma} e^{-d\tau(16\alpha+11)} e^{-\frac{1}{2}dt} \leq 1,$$

which along with (3.13) shows that for all $t \geq \max\{T_4, T_5\}$ and $0 < \lambda \leq \lambda_0$,

$$\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_0)\|_{L^2(I)}^2 + \|v_x(\sigma, \tau - t, \theta_{-\tau}\omega, v_0)\|_{L^2(I)}^2$$

$$\leq 2 + C_{21} \int_{\tau-t}^{\sigma} e^{d(r-\sigma)} \phi_\lambda(r, \theta_{-\tau}\omega) dr$$

$$+ C_{23} e^{-d\sigma} \int_{\tau-t}^{\sigma} e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-\tau-s))} ds \right)^{16\alpha+12} dr$$

$$\leq 2 + C_{21} \int_{-\infty}^{\sigma} e^{d(r-\sigma)} \phi_\lambda(r, \theta_{-\tau}\omega) dr$$

$$+ C_{23} e^{-d\sigma} \int_{-\infty}^{\sigma} e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-\tau-s))} ds \right)^{16\alpha+12} dr.$$

This completes the proof.   □

For later purpose, we prove the following compactness of solutions of problem (2.7)-(2.10).

Lemma 3.3. *Suppose* (2.5) *holds true. Let* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t_n \to \infty$ *and* $(u_{0,n}, e^{\lambda(\omega(-t_n)-\omega(-\tau))} v_{0,n}) \in D(\tau - t_n, \theta_{-t_n}\omega)$ *for some* $D \in \mathcal{D}$. *Then the sequence of the solutions of problem* (2.7)-(2.10),

$$\{(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}), v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}))\}_{n=1}^\infty,$$

has a convergent subsequence in $L^2(I) \times H^1(I)$.

Proof. Since $t_n \to \infty$ and $(u_{0,n}, e^{\lambda(\omega(-t_n)-\omega(-\tau))} v_{0,n}) \in D(\tau - t_n, \theta_{-t_n}\omega)$ for some $D \in \mathcal{D}$, by Lemmas 3.1 and (3.2) with $\sigma = \tau - 1$, we find that there exists $N = N(\tau, \omega, D, \lambda) > 0$ such that for all $n \geq N$,

$$\|u(\tau-1, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\|_{L^2(I)}^2 + \|v(\tau-1, \tau-t_n, \theta_{-\tau}\omega, v_{0,n})\|_{H^1(I)}^2 \leq C_1, \tag{3.14}$$

where $C_1 = C_1(\tau, \omega, \lambda) > 0$. Note that

$$(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}), v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}))$$

$$= (u(\tau, \tau - 1, \theta_{-\tau}\omega, u(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})),$$

$$v(\tau, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}))). \tag{3.15}$$

Then by (3.14)-(3) and Corollary 2.6 we infer that the sequence $\{(u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{0,n}), v(\tau, \tau-t_n, \theta_{-\tau}\omega, v_{0,n}))\}_{n=1}^\infty$ has a convergent subsequence in $L^2(I) \times H^1(I)$. This completes the proof.   □

4. **Existence of tempered random attractors.** This section is devoted to the existence of tempered random attractors for system (2.1)-(2.4). We first present the existence of pullback absorbing sets for the system in $L^2(I) \times H^1(I)$.

Lemma 4.1. *Suppose* (2.5) *holds true. Given* $\lambda > 0$, $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *let*

$$K_\lambda(\tau, \omega) = \{(u, \rho) \in H : \|(u, \rho)\|^2_{L^2(I) \times H^1(I)} \leq L_\lambda(\tau, \omega)\}, \tag{4.1}$$

where $L_\lambda(\tau, \omega)$ *is given by*

$$L_\lambda(\tau, \omega) = L_3(1 + e^{-2\lambda\omega(-\tau)}) + L_3 \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{-\lambda\omega(-s)} ds \right)^2$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r-s))} ds \right)^{16\alpha + 12} dr$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \phi_\lambda(r + \tau, \theta_{-\tau}\omega) dr, \tag{4.2}$$

where L_3 *is a positive constant independent of* τ, ω *and* λ, *and* ϕ_λ *is given by* (3.5). *Then* $K_\lambda = \{K_\lambda(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *is a closed measurable* $\mathcal{D}$-*pullback absorbing set of the cocycle* Φ.

Proof. As a first step, we show that K_λ pullback absorbs every $D \in \mathcal{D}$. By (2.45) we have

$$\rho(\tau, \tau - t, \theta_{-\tau}\omega, \rho_0) = e^{-\lambda\omega(-\tau)} v(\tau, \tau - t, \theta_{-\tau}\omega, v_0), \quad \rho_0 = e^{\lambda(\omega(-t) - \omega(-\tau))} v_0. \tag{4.3}$$

Let $(u_0, \rho_0) \in D(\tau - t, \theta_{-t}\omega)$. Then by (4.3) we find that $(u_0, e^{\lambda(\omega(-t) - \omega(-\tau))} v_0) \in D(\tau - t, \theta_{-t}\omega)$. Thus by Lemma 3.1, there exists $T_1 = T_1(\tau, \omega, D, \lambda) > 0$ such that for all $t \geq T_1$,

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2_{H^1(I)} \leq 2L_1^2 + 2L_1^2 \left(\int_{-\infty}^0 (-s)^{-\frac{7}{8}} e^{ds} e^{\lambda(\omega(-\tau) - \omega(s))} ds \right)^2, \tag{4.4}$$

where L_1 is a positive constant independent of τ, ω, D and λ. By (4.4) we get, for all $t \geq T_1$,

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2_{H^1(I)} \leq 2L_1^2 + 2L_1^2 \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(-s))} ds \right)^2. \tag{4.5}$$

By (4.3) and (4.5) we obtain, for all $t \geq T_1$,

$$\|\rho(\tau, \tau - t, \theta_{-\tau}\omega, \rho_0)\|^2_{H^1(I)}$$

$$\leq 2L_1^2 e^{-2\lambda\omega(-\tau)} + 2L_1^2 e^{-2\lambda\omega(-\tau)} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(-s))} ds \right)^2. \tag{4.6}$$

On the other hand, by Lemma 3.2, there exists $T_2 = T_2(\tau, \omega, D, \lambda) \geq T_1$ such that for all $t \geq T_2$,

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2_{L^2(I)} \leq L_2 + L_2 \int_{-\infty}^0 e^{dr} \phi_\lambda(r + \tau, \theta_{-\tau}\omega) dr$$

$$+ L_2 \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r-s))} ds \right)^{16\alpha + 12} dr, \tag{4.7}$$

where L_2 is a positive constant independent of τ, ω, D and λ, and ϕ_λ is given by (3.5). Let $L_3 = \max\{L_2, 2L_1^2\}$. It follows from (4)-(4.7) that for all $t \geq T_2$ and $(u_0, \rho_0) \in D(\tau - t, \theta_{-t}\omega)$,

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|_{L^2(I)}^2 + \|\rho(\tau, \tau - t, \theta_{-\tau}\omega, \rho_0)\|_{H^1(I)}^2 \leq L_\lambda(\tau, \omega),$$

where $L_\lambda(\tau, \omega)$ is given by (4.1). This shows that for all $t \geq T_2$ and $(u_0, \rho_0) \in D(\tau - t, \theta_{-t}\omega)$,

$$(u(\tau, \tau - t, \theta_{-\tau}\omega, u_0), \rho(\tau, \tau - t, \theta_{-\tau}\omega, \rho_0)) \in K_\lambda(\tau, \omega), \tag{4.8}$$

where $K_\lambda(\tau, \omega)$ is given by (4.1). On the other hand, by (2) we have

$$\Phi(t, \tau - t, \theta_{-t}\omega, (u_0, \rho_0)) = (u(\tau, \tau - t, \theta_{-\tau}\omega, u_0), \rho(\tau, \tau - t, \theta_{-\tau}\omega, \rho_0)). \tag{4.9}$$

By (4.8)-(4.9) we find that $\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_\lambda(\tau, \omega)$ for all $t \geq T_2$, and hence K_λ pullback absorbs every member D of $\mathcal{D}$. Since $L_\lambda(\tau, \omega)$ is measurable in $\omega \in \Omega$, we see that $K_\lambda(\tau, \omega)$ is a closed measurable random set in H.

It remains to show K_λ is tempered, i.e., $K_\lambda \in \mathcal{D}$. Replacing τ by $\tau - t$ and ω by $\theta_{-t}\omega$ in (4.1), after simple calculations, we get

$$L_\lambda(\tau - t, \theta_{-t}\omega) = L_3(1 + e^{-2\lambda(\omega(-\tau) - \omega(-t))})$$

$$+ L_3\left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-t) - \omega(-s-t))} ds\right)^2$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r-s-t))} ds\right)^{16\alpha+12} dr$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \phi_\lambda(r + \tau - t, \theta_{-\tau}\omega) dr, \tag{4.10}$$

where $\phi_\lambda(r + \tau - t, \theta_{-\tau}\omega)$ is given by

$$\phi_\lambda(r + \tau - t, \theta_{-\tau}\omega) = e^{16(\alpha+1)\lambda(\omega(r-t) - \omega(-\tau))} + e^{\frac{4\lambda(\alpha+1)(4\alpha+3)}{2\alpha+1}(\omega(r-t) - \omega(-\tau))}$$

$$+ e^{\frac{\lambda(8\alpha+6)}{\alpha}(\omega(r-t) - \omega(-\tau))} + e^{12\lambda(\omega(r-t) - \omega(-\tau))} + e^{-3\lambda(\omega(r-t) - \omega(-\tau))} + 1.$$

Let $C > 0$ be an arbitrary constant. Then we get from (4.1) and (4) that

$$e^{-2Ct} \|K_\lambda(\tau - t, \theta_{-t}\omega)\|^2 = e^{-2Ct} L_\lambda(\tau - t, \theta_{-t}\omega)$$

$$\leq L_3 e^{-2Ct}(1 + e^{-2\lambda(\omega(-\tau) - \omega(-t))}) + L_3 e^{-2Ct}\left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-t) - \omega(-s-t))} ds\right)^2$$

$$+ L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r-s-t))} ds\right)^{16\alpha+12} dr$$

$$+ L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \phi_\lambda(r + \tau - t, \theta_{-\tau}\omega) dr. \tag{4.11}$$

Next, we show that the right-hand side of (4) converges to zero as $t \to \infty$. Note that for every $\varepsilon > 0$ and $\omega \in \Omega$, there exists $T_0 = T_0(\varepsilon, \omega) > 0$ such that for all $\xi \geq T_0$,

$$|\omega(-\xi)| \leq \varepsilon\xi. \tag{4.12}$$

Let

$$\varepsilon = \min\{\frac{d}{2\lambda}, \frac{d}{\lambda(32\alpha + 24)}, \frac{C}{\lambda(16\alpha + 12)}\}. \tag{4.13}$$

Now for $t \geq T_0$, $s \geq 0$ and $r \leq 0$, we have $t + s - r \geq T_0$, and hence by (4.12) for $\xi = t + s - r$,

$$|\omega(r - s - t)| \leq \varepsilon(t + s - r). \tag{4.14}$$

By (4.14) we have the following estimate for the third term on the right-hand side of (4),

$$L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau) - \omega(r-s-t))} ds \right)^{16\alpha+12} dr$$

$$\leq L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda\omega(-\tau)} e^{\lambda\varepsilon(s+t-r)} ds \right)^{16\alpha+12} dr$$

$$\leq L_3 e^{(16\alpha+12)\lambda\omega(-\tau)} e^{-2Ct} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda\varepsilon s} ds \right)^{16\alpha+12} \int_{-\infty}^0 e^{dr} e^{(16\alpha+12)\lambda\varepsilon(t-r)} dr$$

which along with (4.13) implies

$$L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-s-t))} ds \right)^{16\alpha+12} dr$$

$$\leq L_3 e^{(16\alpha+12)\lambda\omega(-\tau)} e^{-Ct} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-\frac{1}{2}ds} ds \right)^{16\alpha+12} \int_{-\infty}^0 e^{\frac{1}{2}dr} dr$$

$$\leq 2L_3 d^{-1} e^{(16\alpha+12)\lambda\omega(-\tau)} e^{-Ct} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-\frac{1}{2}ds} ds \right)^{16\alpha+12}.$$

Therefore, we obtain that for any $C > 0$,

$$\lim_{t \to \infty} L_3 e^{-2Ct} \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{\lambda(\omega(-\tau)-\omega(r-s-t))} ds \right)^{16\alpha+12} dr = 0. \quad (4.15)$$

Similarly, by (4.13), one can verify that the other terms on the right-hand side of (4) also converge to zero as $t \to \infty$, which together with (4) and (4.15) yields

$$\lim_{t \to \infty} e^{-2Ct} \|K_\lambda(\tau - t, \theta_{-t}\omega)\|^2 = 0.$$

In other words, $K \in \mathcal{D}$. This completes the proof. $\qquad \square$

Next, we prove the $\mathcal{D}$-pullback asymptotic compactness of Φ in H.

Lemma 4.2. *Suppose* (2.5) *holds true. Then for every* $\lambda > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *the sequence* $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, (u_{0,n}, \rho_{0,n}))$ *has a convergent subsequence in* H *whenever* $t_n \to \infty$ *and* $(u_{0,n}, \rho_{0,n}) \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. By (4.3) we have

$$\rho(\tau, \tau - t_n, \theta_{-\tau}\omega, \rho_{0,n}) = e^{-\lambda\omega(-\tau)} v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}), \rho_{0,n} = e^{\lambda(\omega(-t_n)-\omega(-\tau))} v_{0,n}. \tag{4.16}$$

Since $(u_{0,n}, \rho_{0,n}) \in D(\tau - t_n, \theta_{-t_n}\omega)$, by (4.16) we find that $(u_{0,n}, e^{\lambda(\omega(-t_n)-\omega(-\tau))} v_{0,n}) \in D(\tau - t_n, \theta_{-t_n}\omega)$. Therefore, by Lemma 3.3 we know that the sequence

$$(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}), \ v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}))$$

has a convergent subsequence in $L^2(I) \times H^1(I)$, which along with (4.16) show that the sequence

$$(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}), \ \rho(\tau, \tau - t_n, \theta_{-\tau}\omega, \rho_{0,n}))$$

has a convergent subsequence in $L^2(I) \times H^1(I)$. Then by (4.9) we conclude that $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, (u_{0,n}, \rho_{0,n}))$ has a convergent subsequence in H. $\qquad \square$

We are now ready to present the main result of this section as given below.

Theorem 4.3. *Suppose* (2.5) *holds true. Then the cocycle* Φ *of problem* (2.1)-(2.4) *has a unique* $\mathcal{D}$-*pullback attractor* $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \ \omega \in \Omega\} \in \mathcal{D}$ *in* H. *If, in addition, the function* $c : \mathbb{R} \to \mathbb{R}^+$ *is periodic with period* $T > 0$, *then the attractor* $\mathcal{A}$ *is also periodic with period* T, *i.e.,* $\mathcal{A}(\tau + T, \omega) = \mathcal{A}(\tau, \omega)$ *for all* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$.

Proof. By Lemmas 4.1 and 4.2, we obtain the existence and uniqueness of the $\mathcal{D}$-pullback attractor $\mathcal{A}$ immediately from [38, 40]. If the function $c : \mathbb{R} \to \mathbb{R}^+$ is periodic with period $T > 0$, then so is the cocycle Φ, i.e., $\Phi(t, \tau + T, \omega, (u_0, \rho_0)) = \Phi(t, \tau, \omega, (u_0, \rho_0))$ for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_0, \rho_0) \in H$. Then the T-periodicity of $\mathcal{A}$ follows from Proposition 3.2 in [38]. $\qquad\qquad\square$

5. **Convergence of tempered random attractors.** In this section, we investigate the limiting behavior of the solutions of the stochastic system (2.1)-(2.4) as the intensity λ of noise approaches zero. We will show that the $\mathcal{D}$-pullback random attractors of the stochastic system converge to that of a deterministic system in terms of the Hausdorff semi-distance in $L^2(I) \times H^1(I)$ as $\lambda \to 0$.

To indicate the dependence of solutions on λ, from now on, we write the solution of problem (2.1)-(2.4) as $(u_\lambda, \rho_\lambda)$, and the corresponding cocycle as Φ_λ. For the same reason, we write the solution of system (2.7)-(2.10) as (u_λ, v_λ):

$$\frac{\partial u_\lambda}{\partial t} = a \frac{\partial^2 u_\lambda}{\partial x^2} - \frac{\partial}{\partial x}\left(u_\lambda \frac{\partial}{\partial x} f(e^{\lambda \omega(t)} v_\lambda)\right), \quad t > \tau, \tag{5.1}$$

$$\frac{\partial v_\lambda}{\partial t} = b \frac{\partial^2 v_\lambda}{\partial x^2} - d v_\lambda + c(t) e^{-\lambda \omega(t)} u_\lambda, \quad t > \tau, \tag{5.2}$$

with boundary conditions

$$\frac{\partial u_\lambda}{\partial x}(a_1, t) = \frac{\partial u_\lambda}{\partial x}(b_1, t) = \frac{\partial v_\lambda}{\partial x}(a_1, t) = \frac{\partial v_\lambda}{\partial x}(b_1, t) = 0, \quad t > \tau, \tag{5.3}$$

and initial conditions

$$u_\lambda(x, \tau) = u_{0,\lambda}(x), \quad v_\lambda(x, \tau) = v_{0,\lambda}(x). \tag{5.4}$$

In the limiting case $\lambda = 0$, the stochastic system (2.1)-(2.4) becomes a deterministic system:

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(u \frac{\partial}{\partial x} f(\rho)\right), \tag{5.5}$$

$$\frac{\partial \rho}{\partial t} = b \frac{\partial^2 \rho}{\partial x^2} + c(t) u - d\rho, \tag{5.6}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(a_1, t) = \frac{\partial u}{\partial x}(b_1, t) = \frac{\partial \rho}{\partial x}(a_1, t) = \frac{\partial \rho}{\partial x}(b_1, t) = 0, \tag{5.7}$$

and initial conditions

$$u(x, \tau) = u_0(x), \quad \rho(x, \tau) = \rho_0(x), \tag{5.8}$$

In the rest of this paper, we always assume $\lambda \in [0, 1]$. Note that all uniform estimates obtained in the previous sections are valid for $\lambda = 0$. This indicates that system (5.5)-(5.8) is well-posed in H. Let Φ_0 be the corresponding continuous deterministic cocycle associated with problem (5.5)-(5.8). Denote by $\mathcal{D}_0$ the collection of tempered families of deterministic nonempty subsets of H, i.e.,

$$\mathcal{D}_0 = \{D = \{D(\tau) \subseteq H : \tau \in \mathbb{R}\} : \lim_{t \to -\infty} e^{Ct} \|D(\tau + t)\| = 0, \ \forall \tau \in \mathbb{R}, \ \forall \, C > 0\}.$$

By Theorem 4.3, we see that, for every positive λ, Φ_λ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_\lambda \in \mathcal{D}$. By the same argument, we can prove that Φ_0 also has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$. The goal of this section is to investigate the relation between $\mathcal{A}_\lambda$ and $\mathcal{A}_0$ as $\lambda \to 0$.

For $0 < \lambda \leq 1$, let K_λ be the $\mathcal{D}$-pullback absorbing set of Φ_λ given by (4.1). When $\lambda = 0$, we define K_0 to be the following family of subsets of H:

$$K_0 = \left\{ K_0(\tau) = \{(u, \rho) \in H : \|(u, \rho)\|^2_{L^2(I) \times H^1(I)} \leq L_0(\tau)\} : \ \tau \in \mathbb{R} \right\}, \qquad (5.9)$$

where $L_0(\tau)$ is given by

$$L_0(\tau) = 2L_3 + L_3 \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} ds \right)^2$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} ds \right)^{16\alpha+12} dr + 6L_3 \int_{-\infty}^0 e^{dr} dr, \qquad (5.10)$$

with L_3 being the same positive constant as in (4.1). Since Lemma 4.1 is also valid for $\lambda = 0$, we find that K_0 is a $\mathcal{D}_0$-pullback absorbing set of Φ_0 in H. In addition, by (4.1)-(4.1) and (5.9)-(5), one can verify that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\lambda \to 0} \|K_\lambda(\tau, \omega)\|^2 = \lim_{\lambda \to 0} \|L_\lambda(\tau, \omega)\| = \|L_0(\tau)\| = \|K_0(\tau)\|^2. \qquad (5.11)$$

Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$B(\tau, \omega) = \{(u, \rho) \in H : \|(u, \rho)\|^2_{L^2(I) \times H^1(I)} \leq L(\tau, \omega)\}, \qquad (5.12)$$

where $L(\tau, \omega)$ is given by

$$L(\tau, \omega) = L_3(1 + e^{2|\omega(-\tau)|}) + L_3 \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{|\omega(-s)|} ds \right)^2$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \left(\int_0^\infty s^{-\frac{7}{8}} e^{-ds} e^{|\omega(-\tau)|+|\omega(r-s)|} ds \right)^{16\alpha+12} dr$$

$$+ L_3 \int_{-\infty}^0 e^{dr} \phi(r + \tau, \theta_{-\tau}\omega) dr, \qquad (5.13)$$

and $\phi(r + \tau, \theta_{-\tau}\omega)$ is given by

$$\phi(r + \tau, \theta_{-\tau}\omega) = e^{16(\alpha+1)(|\omega(r)|+|\omega(-\tau)|)} + e^{\frac{4(\alpha+1)(4\alpha+3)}{2\alpha+1}(|\omega(r)|+|\omega(-\tau)|)}$$

$$+ e^{\frac{8\alpha+6}{\alpha}(|\omega(r)|+|\omega(-\tau)|)} + e^{12(|\omega(r)|+|\omega(-\tau)|)} + e^{3(|\omega(r)|+|\omega(-\tau)|)} + 1.$$

By (4.1)-(4.1) and (5.12)-(5) we see that $K_\lambda(\tau, \omega) \subseteq B(\tau, \omega)$ for all $\lambda \in (0, 1]$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Therefore, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\bigcup_{0 < \lambda \leq 1} \mathcal{A}_\lambda(\tau, \omega) \subseteq \bigcup_{0 < \lambda \leq 1} K_\lambda(\tau, \omega) \subseteq B(\tau, \omega). \qquad (5.14)$$

On the other hand, by (5.12)-(5) and Lemma 2.5 we infer that there exists a positive constant $C_1 = C_1(\tau, \omega)$ (independent of λ) such that for all $0 < \lambda \leq 1$ and $(u_0, \rho_0) \in B(\tau - 1, \theta_{-1}\omega)$, the solutions of system (5.1)-(5.4) satisfy

$$\|(u_\lambda(\tau, \tau - 1, \theta_{-\tau}\omega, u_0), v_\lambda(\tau, \tau - 1, \theta_{-\tau}\omega, v_0))\|^2_{H^1(I) \times H^2(I)} \leq C_1, \qquad (5.15)$$

where $v_0 = e^{\lambda(\omega(-\tau)-\omega(-1))}\rho_0$. By (4.3) and (5.15) we get, for all $0 < \lambda \leq 1$ and for all $(u_0, \rho_0) \in B(\tau - 1, \theta_{-1}\omega)$

$$\|(u_\lambda(\tau, \tau - 1, \theta_{-\tau}\omega, u_0), \rho_\lambda(\tau, \tau - 1, \theta_{-\tau}\omega, \rho_0))\|^2_{H^1(I) \times H^2(I)} \leq C_1(1 + e^{2|\omega(-\tau)|}).$$

$$(5.16)$$

By (4.9) and (5.16) we obtain, for all $0 < \lambda \leq 1$ and for all $(u_0, \rho_0) \in B(\tau - 1, \theta_{-1}\omega)$,

$$\|\Phi_\lambda(1, \tau - 1, \theta_{-1}\omega, (u_0, \rho_0))\|^2_{H^1(I) \times H^2(I)} \leq C_1(1 + e^{2|\omega(-\tau)|}). \qquad (5.17)$$

By (5.14) and (5.17) we see that for all $0 < \lambda \leq 1$ and $(u_0, v_0) \in \mathcal{A}_\lambda(\tau - 1, \theta_{-1}\omega)$,

$$\|\Phi_\lambda(1, \tau - 1, \theta_{-1}\omega, (u_0, \rho_0))\|^2_{H^1(I) \times H^2(I)} \leq C_1(1 + e^{2|\omega(-\tau)|}). \qquad (5.18)$$

By the invariance of $\mathcal{A}_\lambda$, we have

$$\Phi_\lambda(1, \tau - 1, \theta_{-1}\omega, \mathcal{A}_\lambda(\tau - 1, \theta_{-1}\omega)) = \mathcal{A}_\lambda(\tau, \omega),$$

which together with (5.18) implies

$$\|(u, \rho)\|^2_{H^1(I) \times H^2(I)} \leq C_1(1 + e^{2|\omega(-\tau)|}), \quad \text{for all } (u, \rho) \in \mathcal{A}_\lambda(\tau, \omega) \text{ with } 0 < \lambda \leq 1.$$
$$(5.19)$$

By (5.19) we find that the set $\bigcup_{0 < \lambda \leq 1} \mathcal{A}_\lambda(\tau, \omega)$ is bounded in $H^1(I) \times H^2(I)$ and hence precompact in $L^2(I) \times H^1(I)$, which will be used to prove the upper semi-continuity of $\mathcal{A}_\lambda$ in $L^2(I) \times H^1(I)$ as $\lambda \to 0$.

Next, we establish the convergence of solutions of system (2.1)-(2.4) as $\lambda \to 0$.

Lemma 5.1. *Suppose* (2.5) *holds true. Let* $(u_{0,\lambda}, v_{0,\lambda}) \in H$ *and* $(u_0, \rho_0) \in H$ *such that*

$$\|(u_{0,\lambda}, v_{0,\lambda})\|_{L^2(I) \times H^1(I)} \leq R \quad and \quad \|(u_0, \rho_0)\|_{L^2(I) \times H^1(I)} \leq R,$$

for some $R > 0$. *Then, for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ *and* $\varepsilon \in (0, 1]$, *there exists a positive number* $\lambda_0 = \lambda_0(\tau, \omega, T, \varepsilon)$ *such that for all* $0 < \lambda \leq \lambda_0$ *and* $t \in [\tau, \tau + T]$, *the solutions of systems* (5.1)-(5.4) *and* (5.5)-(5.8) *satisfy*

$$\|u_\lambda(t, \tau, \omega, u_{0,\lambda}) - u(t, \tau, u_0)\|^2_{L^2(I)} + \|v_\lambda(t, \tau, \omega, v_{0,\lambda}) - \rho(t, \tau, \rho_0)\|^2_{H^1(I)}$$

$$\leq M_4(\|u_{0,\lambda} - u_0\|^2_{L^2(I)} + \|v_{0,\lambda} - \rho_0\|^2_{H^1(I)}) + \varepsilon M_5,$$

where M_4 *and* M_5 *are positive constants depending on* τ, ω, T *and* R, *but independent of* ε *and* λ.

Proof. Let $\kappa = u_\lambda - u$ and $\eta = v_\lambda - \rho$. Then by (5.1)-(5.2) and (5.5)-(5.6) we get

$$\frac{\partial \kappa}{\partial t} = a\frac{\partial^2 \kappa}{\partial x^2} - \frac{\partial}{\partial x}\left(\kappa\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v_\lambda) + u(\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v_\lambda) - \frac{\partial}{\partial x}f(\rho))\right), \quad t > \tau,$$
$$(5.20)$$

$$\frac{\partial \eta}{\partial t} = b\frac{\partial^2 \eta}{\partial x^2} - d\eta + c(t)\left(e^{-\lambda\omega(t)}\kappa + (e^{-\lambda\omega(t)} - 1)u\right), \quad t > \tau, \qquad (5.21)$$

with initial conditions

$$\kappa(x, 0) = \kappa_0(x) = u_{0,\lambda}(x) - u_0(x), \quad \eta(x, 0) = \eta_0(x) = v_{0,\lambda}(x) - \rho_0(x). \quad (5.22)$$

Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1]$, by the continuity of ω, we find that there exists $\lambda_0 = \lambda_0(\tau, \omega, T, \varepsilon) \in (0, 1]$ such that for all $\lambda \in (0, \lambda_0]$ and for all $t \in [\tau, \tau + T]$,

$$|e^{\lambda\omega(t)} - 1| + |e^{-\lambda\omega(t)} - 1| < \varepsilon. \qquad (5.23)$$

By (5.20) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\kappa\|^2_{L^2(I)} + a\|\kappa_x\|^2_{L^2(I)}$$

$$= \int_I \kappa_x\kappa\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v_\lambda)dx + \int_I \kappa_x u(\frac{\partial}{\partial x}f(e^{\lambda\omega(t)}v_\lambda) - \frac{\partial}{\partial x}f(\rho))dx. \qquad (5.24)$$

For the last term on the right-hand side of (5) we have

$$\int_I \kappa_x u \left(\frac{\partial}{\partial x} f(e^{\lambda \omega(t)} v_\lambda) - \frac{\partial}{\partial x} f(\rho) \right) dx$$

$$= \int_I \kappa_x u \left(f'(e^{\lambda \omega(t)} v_\lambda) e^{\lambda \omega(t)} \frac{\partial v_\lambda}{\partial x} - f'(\rho) \rho_x \right) dx$$

$$= \int_I \kappa_x u \left((f'(e^{\lambda \omega(t)} v_\lambda) - f'(\rho)) e^{\lambda \omega(t)} \frac{\partial v_\lambda}{\partial x} + f'(\rho)(e^{\lambda \omega(t)} \frac{\partial v_\lambda}{\partial x} - \rho_x) \right) dx$$

$$= \int_I \kappa_x u \left(f''(s)(e^{\lambda \omega(t)} v_\lambda - \rho) e^{\lambda \omega(t)} \frac{\partial v_\lambda}{\partial x} + f'(\rho)(e^{\lambda \omega(t)} \eta_x + (e^{\lambda \omega(t)} - 1)\rho_x) \right) dx$$

$$= \int_I \kappa_x u (f''(s)(e^{\lambda \omega(t)} \eta + (e^{\lambda \omega(t)} - 1)\rho) e^{\lambda \omega(t)} \frac{\partial v_\lambda}{\partial x} + f'(\rho)(e^{\lambda \omega(t)} \eta_x + (e^{\lambda \omega(t)} - 1)\rho_x))$$

$$= \int_I e^{2\lambda \omega(t)} f''(s) \kappa_x u \eta \frac{\partial v_\lambda}{\partial x} dx + \int_I e^{\lambda \omega(t)} (e^{\lambda \omega(t)} - 1) f''(s) \kappa_x u \rho \frac{\partial v_\lambda}{\partial x} dx$$

$$+ \int_I e^{\lambda \omega(t)} f'(\rho) \kappa_x u \eta_x dx + \int_I (e^{\lambda \omega(t)} - 1) f'(\rho) \kappa_x u \rho_x dx. \tag{5.25}$$

We need to estimate every term on the right-hand side of (5.25). First, by Lemmas 2.2 and 2.3, we find that there exists $C_1 = C_1(\tau, \omega, T, R) > 0$ such that for all $0 < \lambda \leq 1$ and $t \in [\tau, \tau + T]$,

$$\|u_\lambda(t, \tau, \omega, u_{0,\lambda})\|_{L^2(I)}^2 + \|v_\lambda(t, \tau, \omega, v_{0,\lambda})\|_{H^1(I)}^2 \leq C_1, \tag{5.26}$$

$$\int_\tau^{\tau+T} (\|u_\lambda(s, \tau, \omega, u_{0,\lambda})\|_{H^1(I)}^2 + \|v_\lambda(s, \tau, \omega, v_{0,\lambda})\|_{H^2(I)})^2 ds \leq C_1, \tag{5.27}$$

$$\|u(t, \tau, u_0)\|_{L^2(I)}^2 + \|\rho(t, \tau, \rho_0)\|_{H^1(I)}^2 \leq C_1, \tag{5.28}$$

and

$$\int_\tau^{\tau+T} (\|u(s, \tau, u_0)\|_{H^1(I)}^2 + \|\rho(s, \tau, \rho_0)\|_{H^2(I)}^2) ds \leq C_1. \tag{5.29}$$

By (1.6), (5.26) and (5.28) we get, for all $0 < \lambda \leq 1$ and $t \in [\tau, \tau + T]$,

$$\|v_\lambda(t, \tau, \omega, v_{0,\lambda})\|_{C(\bar{I})} + \|\rho(t, \tau, \omega, \rho_0)\|_{C(\bar{I})} \leq C_2, \tag{5.30}$$

for some $C_2 = C_2(\tau, \omega, T, R) > 0$. By (2.5), (5.26), (5.28) and (5.30), for the first term on the right-hand side of (5.25) we have, for all $0 < \lambda \leq 1$ and $t \in [\tau, \tau + T]$,

$$|\int_I e^{2\lambda \omega(t)} f''(s) \kappa_x u \eta \frac{\partial v_\lambda}{\partial x} dx| \leq C_3 \int_I |\kappa_x u \eta \frac{\partial v_\lambda}{\partial x}| dx$$

$$\leq C_3 \|\kappa_x\|_{L^2(I)} \|u\|_{L^4(I)} \|\frac{\partial v_\lambda}{\partial x}\|_{L^4(I)} \|\eta\|_{L^\infty(I)}$$

$$\leq C_4 \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|u\|_{L^2(I)}^{\frac{3}{4}} \|\frac{\partial v_\lambda}{\partial x}\|_{H^1(I)}^{\frac{1}{4}} \|\frac{\partial v_\lambda}{\partial x}\|_{L^2(I)}^{\frac{3}{4}} \|\eta\|_{L^\infty(I)}$$

$$\leq C_5 \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|\frac{\partial v_\lambda}{\partial x}\|_{H^1(I)}^{\frac{1}{4}} \|\eta\|_{H^1(I)}$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + C_6 \|u\|_{H^1(I)}^{\frac{1}{2}} \|\frac{\partial v_\lambda}{\partial x}\|_{H^1(I)}^{\frac{1}{2}} \|\eta\|_{H^1(I)}^2$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + C_7 (1 + \|u\|_{H^1(I)}^2 + \|v_\lambda\|_{H^2(I)}^2) \|\eta\|_{H^1(I)}^2. \tag{5.31}$$

Similarly, we can also obtain

$$|\int_I e^{\lambda \omega(t)} f'(\rho) \kappa_x u \eta_x dx| \leq C_8 \|\kappa_x\|_{L^2(I)} \|u\|_{L^4(I)} \|\eta_x\|_{L^4(I)}$$

$$\leq C_9 \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|u\|_{L^2(I)}^{\frac{3}{4}} \|\eta_x\|_{H^1(I)}^{\frac{1}{4}} \|\eta_x\|_{L^2(I)}^{\frac{3}{4}}$$

$$\leq C_{10} \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|\eta_x\|_{H^1(I)}^{\frac{1}{4}} \|\eta_x\|_{L^2(I)}^{\frac{3}{4}}$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + \frac{1}{4} b \|\eta_{xx}\|_{L^2(I)}^2 + C_{11}(1 + \|u\|_{H^1(I)}^2) \|\eta\|_{H^1(I)}^2. \tag{5.32}$$

For the second term on the right-hand side of (5.25), by (5.23), we get, for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\left| \int_I e^{\lambda\omega(t)}(e^{\lambda\omega(t)} - 1) f''(s) \kappa_x u \rho \frac{\partial v_\lambda}{\partial x} dx \right| \leq \varepsilon C_{12} \|\kappa_x\|_{L^2(I)} \|u\|_{L^4(I)} \left\| \frac{\partial v_\lambda}{\partial x} \right\|_{L^4(I)} \|\rho\|_{L^\infty(I)}$$

$$\leq \varepsilon C_{13} \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|u\|_{L^2(I)}^{\frac{3}{4}} \left\| \frac{\partial v_\lambda}{\partial x} \right\|_{H^1(I)}^{\frac{1}{4}} \left\| \frac{\partial v_\lambda}{\partial x} \right\|_{L^2(I)}^{\frac{3}{4}} \|\rho\|_{H^1(I)}$$

$$\leq \varepsilon C_{14} \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \left\| \frac{\partial v_\lambda}{\partial x} \right\|_{H^1(I)}^{\frac{1}{4}}$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + \varepsilon C_{15}(1 + \|u\|_{H^1(I)}^2 + \|v_\lambda\|_{H^2(I)}^2). \tag{5.33}$$

Similarly, for the last term on the right-hand side of (5.25), by (5.23), we get, for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\left| \int_I (e^{\lambda\omega(t)} - 1) f'(\rho) \kappa_x u \rho_x dx \right| \leq \varepsilon C_{16} \|\kappa_x\|_{L^2(I)} \|u\|_{L^4(I)} \|\rho_x\|_{L^4(I)}$$

$$\leq \varepsilon C_{17} \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|u\|_{L^2(I)}^{\frac{3}{4}} \|\rho_x\|_{H^1(I)}^{\frac{1}{4}} \|\rho_x\|_{L^2(I)}^{\frac{3}{4}}$$

$$\leq \varepsilon C_{18} \|\kappa_x\|_{L^2(I)} \|u\|_{H^1(I)}^{\frac{1}{4}} \|\rho\|_{H^2(I)}^{\frac{1}{4}}$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + \varepsilon C_{19}(1 + \|u\|_{H^1(I)}^2 + \|\rho\|_{H^2(I)}^2). \tag{5.34}$$

It follows from (5.25), and (5.31)-(5.34) that for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\int_I \kappa_x u \left(\frac{\partial}{\partial x} f(e^{\lambda\omega(t)} v_\lambda) - \frac{\partial}{\partial x} f(\rho) \right) dx \leq \frac{1}{2} a \|\kappa_x\|_{L^2(I)}^2 + \frac{1}{4} b \|\eta_{xx}\|_{L^2(I)}^2$$

$$+ \varepsilon C_{20}(1 + \|u\|_{H^1(I)}^2 + \|\rho\|_{H^2(I)}^2 + \|v_\lambda\|_{H^2(I)}^2)$$

$$+ C_{21}(1 + \|u\|_{H^1(I)}^2 + \|v_\lambda\|_{H^2(I)}^2) \|\eta\|_{H^1(I)}^2. \tag{5.35}$$

For the first term on the right-hand side of (5) we have

$$\left| \int_I \kappa_x \kappa \frac{\partial}{\partial x} f(e^{\lambda\omega(t)} v_\lambda) dx \right| = \left| \int_I \kappa_x \kappa f'(e^{\lambda\omega(t)} v_\lambda) e^{\lambda\omega(t)} \frac{\partial v_\lambda}{\partial x} dx \right|$$

$$\leq C_{22} \|\kappa_x\|_{L^2(I)} \|\kappa\|_{L^2(I)} \left\| \frac{\partial v_\lambda}{\partial x} \right\|_{L^\infty(I)}$$

$$\leq \frac{1}{8} a \|\kappa_x\|_{L^2(I)}^2 + C_{23} \|v_\lambda\|_{H^2(I)}^2 \|\kappa\|_{L^2(I)}^2. \tag{5.36}$$

By (5) and (5)-(5) we obtain that for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\frac{d}{dt} \|\kappa\|_{L^2(I)}^2 + \frac{3}{4} a \|\kappa_x\|_{L^2(I)}^2$$

$$\leq \frac{1}{2} b \|\eta_{xx}\|_{L^2(I)}^2 + 2\varepsilon C_{20}(1 + \|u\|_{H^1(I)}^2 + \|\rho\|_{H^2(I)}^2 + \|v_\lambda\|_{H^2(I)}^2)$$

$$+ C_{24}(1 + \|u\|_{H^1(I)}^2 + \|v_\lambda\|_{H^2(I)}^2)(\|\kappa\|_{L^2(I)}^2 + \|\eta\|_{H^1(I)}^2). \tag{5.37}$$

On the other hand, by (5.21), (5.23), (5.26)-(5.28) we get

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{H^1(I)}^2 + b(\|\eta_x\|_{L^2(I)}^2 + \|\eta_{xx}\|_{L^2(I)}^2) + d\|\eta\|_{H^1(I)}^2$$

$$= c(t) \int_I e^{-\lambda\omega(t)} \kappa(\eta - \eta_{xx}) dx + c(t) \int_I (e^{-\lambda\omega(t)} - 1) u(\eta - \eta_{xx}) dx.$$

$$\leq C_{25} \|\kappa\|_{L^2(I)} (\|\eta\|_{L^2(I)} + \|\eta_{xx}\|_{L^2(I)}) + \varepsilon C_{26} \|u\|_{L^2(I)} (\|\eta\|_{L^2(I)} + \|\eta_{xx}\|_{L^2(I)})$$

$$\leq \frac{1}{4} b \|\eta_{xx}\|_{L^2(I)}^2 + C_{27} (\|\kappa\|_{L^2(I)}^2 + \|\eta\|_{L^2(I)}^2) + \varepsilon C_{28}. \tag{5.38}$$

By (5)-(5.38) we obtain that for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\frac{d}{dt} (\|\kappa\|_{L^2(I)}^2 + \|\eta\|_{H^1(I)}^2)$$

$$\leq \varepsilon C_{29} (1 + \|u\|_{H^1(I)}^2 + \|\rho\|_{H^2(I)}^2 + \|v_\lambda\|_{H^2(I)}^2)$$

$$+ C_{30} (1 + \|u\|_{H^1(I)}^2 + \|v_\lambda\|_{H^2(I)}^2)(\|\kappa\|_{L^2(I)}^2 + \|\eta\|_{H^1(I)}^2). \tag{5.39}$$

Applying Gronwall's inequality to (5), we obtain, for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\|\kappa(t, \tau, \omega, \kappa_0)\|_{L^2(I)}^2 + \|\eta(t, \tau, \omega, \eta_0)\|_{H^1(I)}^2$$

$$\leq e^{C_{30} \int_\tau^t (1 + \|u(s, \tau, u_0)\|_{H^1(I)}^2 + \|v_\lambda(s, \tau, \omega, v_{0,\lambda})\|_{H^2(I)}^2) ds} (\|\kappa_0\|_{L^2(I)}^2 + \|\eta_0\|_{H^1(I)}^2)$$

$$+ \varepsilon C_{29} \int_\tau^t e^{C_{30} \int_r^t (1 + \|u(s, \tau, u_0)\|_{H^1(I)}^2 + \|v_\lambda(s, \tau, \omega, v_{0,\lambda})\|_{H^2(I)}^2) ds} (1 + \|u(r)\|_{H^1(I)}^2$$

$$+ \|\rho(r)\|_{H^2(I)}^2 + \|v_\lambda(r)\|_{H^2(I)}^2) dr,$$

which together with (5.27) and (5.29) implies that for all $0 < \lambda \leq \lambda_0$ and $t \in [\tau, \tau + T]$,

$$\|\kappa(t, \tau, \omega, \kappa_0)\|_{L^2(I)}^2 + \|\eta(t, \tau, \omega, \eta_0)\|_{H^1(I)}^2 \leq C_{31} (\|\kappa_0\|_{L^2(I)}^2 + \|\eta_0\|_{H^1(I)}^2) + \varepsilon C_{32}. \tag{5.40}$$

By (5.22) and (5.40) we conclude the proof. □

We now present the convergence of the solutions of system (2.1)-(2.4) as $\lambda \to 0$.

Lemma 5.2. *Suppose* (2.5) *holds true. Let* $(u_{0,\lambda}, \rho_{0,\lambda}) \in H$ *and* $(u_0, \rho_0) \in H$ *such that*

$$(u_{0,\lambda}, \rho_{0,\lambda}) \to (u_0, \rho_0) \quad in \quad L^2(I) \times H^1(I) \quad as \quad \lambda \to 0.$$

Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ *and* $t \in [\tau, \tau + T]$,

$$(u_\lambda(t, \tau, \omega, u_{0,\lambda}), \rho_\lambda(t, \tau, \omega, \rho_{0,\lambda})) \to (u(t, \tau, u_0), \rho(t, \tau, \rho_0)) \quad in \quad L^2(I) \times H^1(I)$$

as $\lambda \to 0$, *where* $(u_\lambda(t, \tau, \omega, u_{0,\lambda}), \rho_\lambda(t, \tau, \omega, \rho_{0,\lambda}))$ *and* $(u(t, \tau, u_0), \rho(t, \tau, \rho_0))$ *are the solutions of system* (2.1)-(2.4) *and system* (5.5)-(5.8), *respectively.*

Proof. This follows from (2.45) and Lemma 5.1 immediately. □

We finally prove the upper semi-continuity of $\mathcal{D}$-pullback random attractors for the stochastic system (2.1)-(2.4).

Theorem 5.3. *Suppose* (2.5) *holds true. Then for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\lambda \to 0} \text{dist}_{L^2(I) \times H^1(I)} (\mathcal{A}_\lambda(\tau, \omega), \mathcal{A}_0(\tau)) = 0. \tag{5.41}$$

Proof. Based on (5.11), (5.19) and Lemma 5.2, we see that (5.41) follows from Theorem 3.2 in [39] directly. □

REFERENCES

[1] A. Adili and B. Wang, Random attractors for stochastic FitzHugh-Nagumo systems driven by deterministic non-autonomous forcing, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 643–666.

[2] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[3] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246** (2009), 845–869.

[4] W. J. Beyn, B. Gess, P. Lescot and M. Röckner, The global random attractor for a class of stochastic porous media equations, *Comm. Partial Differential Equations*, **36** (2011), 446–469.

[5] T. Caraballo, M. Garrido-Atienza, B. Schmalfuss and J. Valero, Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Contin. Dyn. Syst.*, **21** (2008), 415–443.

[6] T. Caraballo, J. Real and I. Chueshov, Pullback attractors for stochastic heat equations in materials with memory, *Discrete Contin. Dyn. Syst. Ser. B*, **9** (2008), 525–539.

[7] T. Caraballo and J. Langa, On the upper semicontinuity of cocycle attractors for nonautonomous and random dynamical systems, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, **10** (2003), 491–513.

[8] T. Caraballo, M. Garrido-Atienza, B. Schmalfuss and J. Valero, Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 439–455.

[9] T. Caraballo, M. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.*, **74** (2011), 3671–3684.

[10] T. Caraballo, J. Langa, V. S. Melnik and J. Valero, Pullback attractors for nonautonomous and stochastic multivalued dynamical systems, *Set-Valued Analysis*, **11** (2003), 153–201.

[11] I. Chueshov and M. Scheutzow, On the structure of attractors and invariant measures for a class of monotone random systems, *Dynamical Systems*, **19** (2004), 127–144.

[12] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dyn. Diff. Eqns.*, **9** (1997), 307–341.

[13] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Th. Re. Fields*, **100** (1994), 365–393.

[14] J. Duan and B. Schmalfuss, The 3D quasigeostrophic fluid dynamics under random forcing on boundary, *Comm. Math. Sci.*, **1** (2003), 133–151.

[15] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, **59** (1996), 21–45.

[16] H. Gajewski and K. Zacharias, Global behavior of a reaction-diffusion system modeling chemotaxis, *Math. Nachr.*, **195** (1998), 77–114.

[17] M. Garrido-Atienza and B. Schmalfuss, Ergodicity of the infinite dimensional fractional Brownian motion, *J. Dynam. Differential Equations*, **23** (2011), 671–681.

[18] M. Garrido-Atienza, A. Ogrowsky and B. Schmalfuss, Random differential equations with random delays, *Stoch. Dyn.*, **11** (2011), 369–388.

[19] M. Garrido-Atienza, B. Maslowski and B. Schmalfuss, Random attractors for stochastic equations driven by a fractional Brownian motion, *International J. Bifurcation and Chaos*, **20** (2010), 2761–2782.

[20] B. Gess, W. Liu and M. Rockner, Random attractors for a class of stochastic partial differential equations driven by general additive noise, *J. Differential Equations*, **251** (2011), 1225–1253.

[21] B. Gess, Random attractors for degenerate stochastic partial differential equations, *J. Dyn. Diff. Eqns.*, **25** (2013), 121–157.

[22] B. Gess, Random attractors for singular stochastic evolution equations, *J. Differential Equations*, **255** (2013), 524–559.

[23] M. A. Herrero and J. J. L. Velazquez, A blow-up mechanism for a chemotaxis model, *Ann. Scoula. Norm. Sup. Pisa IV*, **24** (1997), 633–683.

[24] D. Horstmann and G. Wang, Blowup in a chemotaxis model without symmetry assumptions, *European J. Appl. Math.*, **12** (2001), 159–177.

[25] J. Huang and W. Shen, Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains, *Discrete Contin. Dyn. Syst.*, **24** (2009), 855–882.

[26] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as instability, *J. Theor. Biol.*, **26** (1970), 399–415.

[27] P. E. Kloeden and J. Langa, Flattening, squeezing and the existence of random attractors, *Proc. Royal Soc. London Serie A.* , **463** (2007), 163–181.

[28] I. R. Lapidus and M. Levandowsky, Modeling chemosensory responses of swimming eukaryotes, biological growth and spread, *Lecture Notes in Biomathematics*, **38** (1980), 388–396, Springer-Verlag, New York.

[29] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, *J. Differential Equations*, **244** (2008), 1–23.

[30] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.*, **40** (1997), 411–433.

[31] T. Nagai, T. Senba and T. Suzuki, Chemotactic collapse in a parabolic system of mathematical biology, *Hiroshima Math. J.*, **30** (2000), 463–497.

[32] K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.*, **44** (2001), 441–469.

[33] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, *Nonlinear Analysis, TMA*, **51** (2002), 119–144.

[34] R. Schaaf, Stationary solutions of chemotaxis systems, *Trans. Amer. Math. Soc.*, **292** (1985), 531–556.

[35] B. Schmalfuss, Backward cocycles and attractors of stochastic differential equations, *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, 185–192, Dresden, 1992.

[36] B. Wang, Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains, *J. Differential Equations*, **246** (2009), 2506–2537.

[37] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, **363** (2011), 3639–3663.

[38] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012), 1544–1583.

[39] B. Wang, Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stoch. Dyn.*, **14** (2014), 1450009, 31pp.

[40] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst. Ser. A*, **34** (2014), 269–300.

[41] A. Yagi, Norm behavior of solutions to the parabolic system of chemotaxis, *Math. Japonica*, **45** (1997), 241–265.

E-mail address: gzydshang@126.com
E-mail address: jtian@nmsu.edu
E-mail address: bwang@nmt.edu