Applicable Analysis, 2014
http://dx.doi.org/10.1080/00036811.2014.918606

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Some results in Floquet theory, with application to periodic epidemic models

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Communicated by C. Mardare

(*Received 24 November 2013; accepted 23 April 2014*)

In this paper, we present several new results to the classical Floquet theory on the study of differential equations with periodic coefficients. For linear periodic systems, the Floquet exponents can be directly calculated when the coefficient matrices are triangular. Meanwhile, the Floquet exponents are eigenvalues of the integral average of the coefficient matrices when they commute with their antiderivative matrices. For the stability analysis of constant and nontrivial periodic solutions of nonlinear differential equations, we derive a few results based on linearization. We also briefly discuss the properties of Floquet exponents for delay linear periodic systems. To demonstrate the application of these analytical results, we consider a new cholera epidemic model with phage dynamics and seasonality incorporated. We conduct mathematical analysis and numerical simulation to the model with several periodic parameters.

Keywords: Floquet exponents; stability; epidemic models

AMS Subject Classifications: 34C25; 34D20; 92D30

1. Introduction

Floquet theory [1] is concerned with the study of the linear stability of differential equations with periodic coefficients. A focus of this theory is the concept of Floquet exponents which are analogous to the eigenvalues of the variational (Jacobian) matrices at equilibrium points. The framework of Floquet theory offers a powerful means to analyze nonautonomous, periodic differential equations. However, the problem of determining the Floquet exponents of general linear periodic systems is difficult.[2] Except for a few special cases, which include second-order scalar equations and systems of Hamiltonian type or canonical forms, very little is known on the analysis of Floquet exponents. Nevertheless, we mention that there have been several numerical studies on the calculation of Floquet multipliers, such as using the time domain methods [3,4] and the frequency domain methods.[5,6]

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In addition, although the dichotomy has been an important concept in stability analysis of nonautonomous systems,[7] Floquet exponents still play a central role in the study of periodic nonautonomous systems.

The present paper aims to make contribution toward partially overcoming the challenge of analyzing Floquet exponents. We prove several properties of linear periodic systems that can facilitate the analysis of Floquet exponents. For example, when the coefficient matrices commute with their antiderivative matrices, the Floquet exponents are eigenvalues of the integral average of the coefficient matrices. Another useful property we find is that the Floquet exponents can be directly computed from diagonal entries when the coefficient matrices are triangular.

For a nonlinear nonautonomous periodic system, if it has a nonconstant periodic solution, its stability can be analyzed by linearization about the periodic solution. The variational system then becomes a linear periodic system, and its Floquet exponents provide useful information on the stability of the periodic solution. We apply our findings in Floquet theory to such studies, and present easy-to-check properties for the stability of periodic solutions. In addition, we briefly discuss delay linear periodic systems. Generally, such a system has (countably) infinitely many Floquet exponents, and it is impossible to explicitly calculate all the Floquet exponents. Nevertheless, when the coefficient matrices are triangular, we can present clear relationship between Floquet exponents of a delay linear periodic system and those of the corresponding nondelay system. This is useful for comparison of the delay effects.

This work is motivated by the study of deterministic (i.e. differential equation based) epidemic models with seasonal oscillations. Currently, most epidemic models in literature assume the system parameters (i.e. those coefficients typically representing various rates related to disease transmission) are constants, which leads to simplified dynamics. Stability analysis of such dynamical systems has been extensively conducted and has become standard practice for prototype models such as SIS, SIR, and SEIR.[8–10] Practically, however, seasonal variations and climatic events may have significant impacts on the disease transmission rates. Hence, these parameters may vary with time and exhibit periodic oscillations, which directly affect the system dynamics. It is thus important to investigate these models with time-periodic parameters to improve our understanding of the complex mechanism in the initiation and spread of infectious diseases.

As a specific application of our findings in Floquet theory, we study the mathematical modeling of cholera, an infectious disease characterized by severe diarrhea and caused by the bacterium *Vibrio cholerae* in contaminated water and food. Cyclical pattern of cholera outbreaks in time has long been observed.[11–13] We propose a new cholera epidemic model in this paper which incorporates the bacteria-phage interaction and seasonal variation. We then apply our results in Floquet theory to analyze this periodic model with several time-periodic parameters. Numerical simulation results are also presented to verify the analytical predictions.

In what follows, we will first present a brief review of Floquet theory and highlight several new results in Section 2. In Section 3, we will apply our findings to analyze the new cholera model in periodic environments. Finally, we conclude the paper with some discussion.

2. Floquet theory

2.1. *Linear periodic systems*

Floquet exponents or characteristic multipliers (see Definition 2.1) are defined in terms of a fundamental matrix solution of nonautonomous linear systems. The difficulty is how to determine the Floquet exponents without knowing explicit solutions. In fact, for most nonautonomous linear systems, analytical solutions in explicit form are impossible to obtain. In this section, we shall recall the Floquet theorem and related basic properties. We then give the formulae to compute Floquet exponents directly for two types of linear periodic systems without knowing their solutions. When the coefficient matrix $A(t)$ commutes with its antiderivative matrix $\int_0^t A(\tau)d\tau$, the Floquet exponents of the system $\dot{X} = A(t)X$ are eigenvalues of a constant matrix. When $A(t)$ is a triangular matrix, the Floquet exponents of the system $\dot{X} = A(t)X$ can be computed by the diagonal entries of $A(t)$.

Consider the homogeneous linear periodic system

$$\dot{X} = A(t)X, \quad A(t+\omega) = A(t), \quad \omega > 0, \quad X \in R^n, \tag{1}$$

where $A(t)$ is a continuous $n \times n$ real matrix-valued function of t. For completeness, we briefly review the Floquet theorem and some basic properties below. For more details of Floquet theory, we refer to [2,14–16].

It will help our discussion by first introducing the concept of matrix logarithm. For a square matrix A, if the absolute values of its eigenvalues are less than 1, we can define $\log(I + A) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{A^n}{n}$, where I is the identity matrix. The series is convergent. By substitution, it is easy to verify that $e^B = I + A$, where $B = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{A^n}{n}$. For a nonsingular square matrix M, denote its eigenvalue with the maximum absolute value by $\rho(M)$. If all the eigenvalues have the same sign, we define $\log M = (\log \rho(M))I + \log(I + \frac{M-\rho(M)I}{\rho})$. It is also easy to see $\log(P^{-1}MP) = P^{-1}(\log M)P$ for any nonsingular matrix P, and $\log(M_1 \oplus M_2) = \log M_1 \oplus \log M_2$ provided that $\log M_1$ and $\log M_2$ are defined. If the eigenvalues of M have different signs, there exists a nonsingular matrix P such that $P^{-1}MP = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = M_1 \oplus M_2$ and the eigenvalues of each factor matrix have the same sign. Hence, $\log M$ can be defined for any nonsingular matrix. By this definition, we have $\log M = R + \sqrt{-1}(\mathrm{Arg}\rho(M) + 2k\pi)I$ for a given nonsingular matrix M, where R is a real matrix. Thus, $\log M$ is not unique, and these $\log M$'s may differ by $2k\pi\sqrt{-1}I$, where k is an integer. However, if M is a nonsingular positive matrix or has a positive $\rho(M)$, and $\log M$ is required to be a real matrix, then $\log M$ is uniquely determined.

Theorem 2.1 [1] *Every fundamental matrix solution $\Phi(t)$ of the system (1) has the form*

$$\Phi(t) = P(t)e^{Bt}$$

where $P(t)$, B are $n \times n$ matrices, $P(t+\omega) = P(t)$ for all t, and B is a constant matrix.

Corollary 2.1 [17] *There exists a nonsingular periodic transformation of variables which transforms the system (1) into a system with constant coefficients. Actually if $X = P(t)Y$, then the system (1) becomes $\dot{Y} = BY$.*

Definition 2.1 A *monodromy matrix* of the system (1) is a nonsingular matrix C associated with a fundamental matrix solution $\Phi(t)$ of the system (1) through the relation $\Phi(t+\omega) = \Phi(t)C$. The eigenvalues of a monodromy matrix are called the *characteristic multipliers* of the system (1) and the eigenvalues of $\frac{1}{\omega}\log C$ are called *characteristic exponents* or *Floquet exponents* of the system (1).

For different fundamental matrix solutions of the same linear system, the monodromy matrices associated with them are similar to each other. Thus, the characteristic multipliers are unique, and the real parts of the Floquet exponents are also uniquely defined.

Theorem 2.1 states that any solution of the system (1) is a linear combination of functions of the form $p(t)e^{\lambda t}$, where $p(t)$ is a periodic function. Furthermore, Lemma 2.1 in the following states the condition when a complex number can be a Floquet exponent, whereas Lemma 2.2 gives a relation between the sum of all Floquet exponents and the trace of the coefficient matrix of the system (1).

LEMMA 2.1 *A complex number λ is a Floquet exponent of the system (1) if and only if there is a nontrivial solution of the system (1) of the form $p(t)e^{\lambda t}$ where $p(t+\omega) = p(t)$. Particularly, there is a periodic solution of the system (1) of period ω (or 2ω) if and only if there is a characteristic multiplier $\rho = 1$ (or -1).*

LEMMA 2.2 *If $\rho_j = e^{\lambda_j \omega}$, $j = 1, 2, \cdots, n$, are the characteristic multipliers of the system (1), then*

$$\prod_{j=1}^{n} \rho_j = \exp\left(\int_0^{\omega} trA(s)ds\right),$$

$$\sum_{j=0}^{n} \lambda_j = \frac{1}{\omega}\int_0^{\omega} trA(s)ds \left(mod \frac{2\pi i}{\omega}\right).$$

Essentially, Floquet theory converts the study of stability of the trivial solution of the system (1) to a linear system with constant coefficients, $\dot{Y} = BY$. The following theorem provides details about the stability of the system (1) in terms of Floquet exponents. For reference, we refer to Theorem 7.1 on page 120 of Hale's book [2].

THEOREM 2.2 *(i) A necessary and sufficient condition that the system (1) is uniformly stable is that the characteristic multipliers of the system (1) have moduli ≤ 1 (the Floquet exponents have real parts ≤ 0) and the ones with moduli $= 1$ (the Floquet exponents with real parts $= 0$) have multiplicity 1. (ii) A necessary and sufficient condition that the system (1) is uniformly asymptotically stable is that all characteristic multipliers of the system (1) have moduli < 1 (all Floquet exponents have real parts < 0).*

At first glance, it might appear that linear periodic systems share the same simplicity as linear systems with constant coefficients. However, there is an important difference — the characteristic multipliers or Floquet exponents are defined based on the solutions of the system (1), and there is no obvious relation between Floquet exponents and the matrix $A(t)$. Below, for two types of coefficient matrices $A(t)$, we give formulae for directly computing their Floquets without solving the differential equations.

Suppose $A(t)$ is a real and differentiable matrix function of t. When $\frac{dA(t)}{dt}A(t) = A(t)\frac{dA(t)}{dt}$, where the derivative is taken entry-wise, we define

$$e^{A(t)} = I + A(t) + \frac{1}{2!}A^2(t) + \frac{1}{3!}A^3(t) + \cdots.$$

It is easy to check that $\frac{d}{dt}e^{A(t)} = e^{A(t)}\frac{dA(t)}{dt} = \frac{dA(t)}{dt}e^{A(t)}$.

THEOREM 2.3 *Given the homogeneous linear periodic system $\dot{X} = A(t)X$, $A(t+\omega) = A(t)$. If $A(t)\int_0^t A(\tau)d\tau = (\int_0^t A(\tau)d\tau)A(t)$, then a monodromy matrix is given by $C = \exp(\int_0^\omega A(t)dt)$, and the Floquet exponents can be selected as the eigenvalues of the matrix $B = \frac{1}{\omega}\int_0^\omega A(t)dt$.*

Proof Since $A(t)\int_0^t A(\tau)d\tau = (\int_0^t A(\tau)d\tau)A(t)$, the matrix function $\Phi(t) = exp(\int_0^t A(\tau)d\tau)$ is well-defined. It also implies $\int_0^\omega A(t)dt \int_0^t A(\tau)d\tau = (\int_0^t A(\tau)d\tau)\int_0^\omega A(t)dt$. Then $\det\Phi(t) \neq 0$ for $t \geq 0$. The set of column vectors of $\Phi(t)$ are linearly independent. Since $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(t)$ is a fundamental matrix solution of $\dot{X} = A(t)X$.

$$\begin{aligned}\Phi(t+\omega) = e^{\int_0^{t+\omega} A(\tau)d\tau} &= e^{\int_0^t A(\tau)d\tau}e^{\int_t^{t+\omega} A(\tau)d\tau}\\ &= e^{\int_0^t A(\tau)d\tau}e^{\int_0^\omega A(\tau)d\tau}\\ &= \Phi(t)C,\end{aligned}$$

where $C = e^{\int_0^\omega A(t)dt} = e^{B\omega}$, and $B = \frac{1}{\omega}\int_0^\omega A(t)dt$. □

For a diagonal matrix $A(t)$, it commutes with its antiderivative matrix. So we have the following corollary.

COROLLARY 2.2 *If $A(t)$ is a diagonal matrix, $A(t) = \mathrm{diag}\{a_1(t), a_2(t), \cdots, a_n(t)\}$, the Floquet exponents are given by $\frac{1}{\omega}\int_0^\omega a_1(t)dt$, $\frac{1}{\omega}\int_0^\omega a_2(t)dt$, $\cdots$, $\frac{1}{\omega}\int_0^\omega a_n(t)dt$.*

When the system is two-dimensional, there are a few special cases for which the matrix function commutes with its antiderivative.

COROLLARY 2.3 *For a 2×2 matrix function $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{pmatrix}$, if one of the following conditions is satisfied, then $A(t)\int_0^t A(\tau)d\tau = (\int_0^t A(\tau)d\tau)A(t)$.*

(i) *$a_{12}(t)$ and $a_{21}(t)$ both are nonzero constants, and $a_{11}(t) - a_{22}(t)$ is a constant.*
(ii) *$a_{11}(t) = a_{22}(t)$ are constant, and $a_{12}(t)\int_0^t a_{21}(s)ds = a_{21}(t)\int_0^t a_{12}(s)ds$.*
(iii) *$A(t)$ is a constant matrix.*

Proof Denote $a_{ij} = a_{ij}(t)$ and $\bar{a}_{ij} = \int_0^t a_{ij}(s)ds$. To ensure $A(t)\int_0^t A(\tau)d\tau = (\int_0^t A(\tau)d\tau)A(t)$, we have the following sufficient and necessary conditions:

(1) $a_{12}\bar{a}_{21} = \bar{a}_{12}a_{21}$,
(2) $a_{11}\bar{a}_{12} + a_{12}\bar{a}_{22} = \bar{a}_{11}a_{12} + \bar{a}_{12}a_{22}$,
(3) $a_{11}\bar{a}_{21} + a_{12}\bar{a}_{22} = \bar{a}_{11}a_{21} + \bar{a}_{21}a_{22}$.

We can then easily verify each case. When $A(t)$ is a constant matrix, it is straightforward to see the three commutative conditions are satisfied. When $a_{12}(t)$ and $a_{21}(t)$ both are nonzero constants, the commutative condition (1) holds. The condition (2) becomes $(a_{11}-a_{22})a_{12}t = (\bar{a}_{11} - \bar{a}_{22})a_{12}$. Then we have $(a_{11} - a_{22})t = \int_0^t (a_{11} - a_{22})ds$, which implies $a_{11} - a_{22}$ is a constant. Conversely, if $a_{11}(t) - a_{22}(t)$ is a constant, the commutative conditions (2) and (3) both hold. Case (ii) can be directly verified in a similar way. □

We can also establish the following result on two-dimensional linear periodic systems.

THEOREM 2.4 *Given* $\dot{X} = A(t)X$, $A(t + \omega) = A(t)$, *where*

$$A(t) = \begin{pmatrix} -a_{11}(t) & a_{12}(t) \\ a_{21}(t) & -a_{22}(t) \end{pmatrix},$$

and $a_{ij}(t) > 0$ $(i, j = 1, 2)$*. Then, its characteristic multipliers are distinct and positive real numbers, and its Floquet exponents can be selected as real numbers whose sum is negative.*

Proof Denote the fundamental matrix solution of the given system by $\Phi(t)$ with $\Phi(0) = I$. We will first show that each entry of $\Phi(t)$ is positive for $t \in (0, \omega]$.

Since the off diagonal entries of $A(t)$ are positive, we may choose a γ so that each entry of the matrix $A(t) + \gamma I$ is positive for $t \in (0, \omega]$. If $\Psi(t) = \Phi(t)exp(\gamma t)$, then

$$\dot{\Psi}(t) = (A(t) + \gamma I)\Psi(t), \quad \Psi(0) = I,$$

so that

$$\Psi(t) \equiv I + \int_0^t (A(s) + \gamma I)\Psi(s)ds. \tag{2}$$

Let N be a positive integer such that $\frac{\omega}{N} \max_{s\in[0,\omega]} |A(s) + \gamma I| < 1$. Then we apply the method of successive approximation to the matrix sequence $\Psi_0(t) = I$,

$$\Psi_{n+1}(t) = I + \int_0^t (A(s) + \gamma I)\Psi_n(s)ds, \quad n = 0, 1, 2, \cdots,$$

which tends uniformly to $\Psi(t)$ on $(0, \frac{\omega}{N}]$. By mathematical induction, every entry of $\Psi_n(t)$ is positive, $n = 0, 1, 2, \cdots$. Hence, the same is true for the limit $\Psi(t)$. Therefore, each entry of $\Phi(t) = \Psi(t)\exp(-\gamma t)$ is strictly positive. Repeating this procedure on $[\frac{k\omega}{N}, \frac{(k+1)\omega}{N}]$, $k = 1, 2, \cdots, N - 1$, and replacing the initial value I in (2) by $\Psi(\frac{k\omega}{N})$, we obtain the positivity of each entry of $\Phi(t)$ on $[0, \omega]$.

For the positive matrix $\Phi(\omega)$, which is a monodromy of the system, Perron's Theorem [18] states that it has a simple positive eigenvalue which is strictly greater than the other eigenvalue and has a corresponding eigenvector with positive components. Denoting the characteristic multipliers by ρ_1 and ρ_2, then

$$\rho_1\rho_2 = \det(\Phi(\omega)) = \exp \int_0^\omega TrA(t)dt > 0,$$

and we see that both ρ_1 and ρ_2 are positive real numbers. Hence, the two Floquet exponents λ_1 and λ_2 can be selected as real numbers, and

$$\lambda_1 + \lambda_2 = -\frac{1}{\omega} \int_0^\omega (a_{11}(t) + a_{22}(t))dt < 0,$$

since $a_{11}(t)$ and $a_{22}(t)$ are positive functions. □

Next, we turn to the discussion of linear periodic systems with a triangular matrix $A(t)$, and we show that the Floquet exponents can be directly calculated. We state this result for lower triangular matrix functions. Similarly, the result holds for upper triangular matrix functions. We also assume that $A(t)$ is a continuos matrix function in what follows.

THEOREM 2.5 *If $A(t)$ is a periodic lower triangular matrix function*

$$\begin{pmatrix} a_{11}(t) & 0 & \cdots & 0 \\ a_{21}(t) & a_{22}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix},$$

then the characteristic multipliers of the system $\dot{X} = A(t)X$ are given by $\exp(\int_0^\omega a_{11}(t)dt)$, $\exp(\int_0^\omega a_{22}(t)dt)$, $\cdots$, $\exp(\int_0^\omega a_{nn}(t)dt)$, *and a set of its Floquet exponents are given by*

$$\frac{1}{\omega}\int_0^\omega a_{11}(t)dt, \quad \frac{1}{\omega}\int_0^\omega a_{22}(t)dt, \ \cdots, \quad \frac{1}{\omega}\int_0^\omega a_{nn}(t)dt.$$

Proof We will use the variation-of-constants formula to compute a monodromy matrix. Writing $\dot{X} = A(t)X$ in scalar form, we have

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)x_1, \\ \dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2, \\ &\cdots, \\ \dot{x}_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n. \end{aligned}$$

We can see the first component of a solution to the system is $x_1^1(t) = e^{\int_0^t a_{11}(\tau)d\tau}$ with $x_1^1(0) = 1$. Using the variation-of-constants formula, one has

$$x_2^1(t) = e^{\int_0^t a_{22}(\tau)d\tau} \int_0^t a_{21}(s)e^{-\int_0^s a_{22}(\tau)d\tau} x_1^1(s)ds$$

with $x_2^1(0) = 0$. Generally,

$$x_k^1(t) = e^{\int_0^t a_{kk}(s)ds} \int_0^t \left(\sum_{j=1}^{k-1} a_{kj}(s)x_j^1(s)\right) e^{-\int_0^s a_{kk}(\tau)d\tau} ds, \quad k = 2, 3, \cdots, n.$$

Putting them together as a column vector, we obtain a solution to the system (1), denoted by $X^1(t)$.

Then, we compute a second solution with the first component $x_1 = x_1^2(t) = 0$. Similarly, we solve the system one by one, getting $x_2^2(t) = e^{\int_0^t a_{22}(\tau)d\tau}$, and

$$x_3^2(t) = e^{\int_0^t a_{33}(\tau)d\tau} \int_0^t a_{32}(s)e^{-\int_0^s a_{33}(\tau)d\tau} x_2^2(s)ds,$$

and $x_4^2(t), \cdots, x_n^2(t)$. Putting them together as a column vector, we obtain a second solution to the system (1), denoted by $X^2(t)$.

Next, we calculate a third solution with $x_1 = x_1^3(t) = x_2 = x_2^3(t) = 0$. Using the same method described above, we solve for $x_3, x_4, \cdots, x_n$, and denote these components by $x_3^3(t)$,

$x_4^3(t), \cdots, x_n^3(t)$. Putting them together as a column vector, we obtain another solution to the system (1), denoted by $X^3(t)$.

Repeating this procedure, we obtain a set of linearly independent solutions $X^1(t), \cdots, X^n(t)$, which form a fundamental solution matrix $\Phi(t) = (X^1(t), X^2(t), \cdots, X^n(t))$. It is easy to check that $\Phi(0) = I$ and $\Phi(t)$ is a lower triangular matrix. The diagonal entries of $\Phi(t)$ are given by $\exp(\int_0^t a_{11}(s)ds)$, $\exp(\int_0^t a_{22}(s)ds)$, $\cdots$, $\exp(\int_0^t a_{nn}(s)ds)$. By the definition of a monodromy matrix, $\Phi(t+\omega) = \Phi(t)C$, and the monodromy matrix C must be a lower triangular matrix. The diagonal entries of C are $\exp(\int_0^\omega a_{11}(s)ds)$, $\exp(\int_0^\omega a_{22}(s)ds)$, $\cdots$, $\exp(\int_0^\omega a_{nn}(s)ds)$, which are the characteristic multipliers. Then the Floquet exponents are given by $\frac{1}{\omega}\int_0^\omega a_{11}(t)dt$, $\frac{1}{\omega}\int_0^\omega a_{22}(t)dt$, $\cdots$, $\frac{1}{\omega}\int_0^\omega a_{nn}(t)dt$. $\square$

From the proof of Theorem 2.5, we have the following two corollaries.

Corollary 2.4 *If $A(t)$ is a periodic triangular matrix, then the Floquet exponents of the system $\dot{X} = A(t)X$ can be represented by real numbers.*

Corollary 2.5 *If $A(t)$ is a periodic block triangular matrix, then the system $\dot{X} = A(t)X$ has a block triangular monodromy matrix.*

Denote $\lambda_i = \frac{1}{\omega}\int_0^\omega a_{ii}(t)dt$, $i = 1, 2, \cdots, n$. Define periodic functions $p_i(t) = e^{\int_0^t (a_{ii}(s)-\lambda_i)ds}$, $i = 1, 2, \cdots, n$. Suppose there are n different Floquet exponents; namely, each Floquet exponent has multiplicity 1. Form vector-valued periodic functions $V_1(t) = (p_1(t), p_{21}(t), \cdots, p_{n1}(t))$, $V_2(t) = (0, p_2(t), \cdots, p_{n2}(t))$, $\cdots$, $V_n(t) = (0, 0, \cdots, 0, p_n(t))$, where $p_{ij}(t)$, $i \neq j$ are some periodic functions. These vector-valued functions are considered as column vectors. As in the proof of Theorem 2.5, denote $B = \log C$, then B is a lower triangular matrix. Since the fundamental matrix solution $\Phi(t)$ is lower triangular, then $\Phi(t)e^{-Bt} = P(t)$ is also lower triangular. The diagonal entries of $P(t)$ are $p_i(t)$'s. Thus, we have a proposition about the solutions of triangular systems.

Proposition 2.1 *Let $\dot{X} = A(t)X$, where $A(t)$ is a periodic lower triangular matrix. Then any solution of this system can be expressed in the form*

$$k_1(t)e^{\lambda_1 t}V_1(t) + k_2(t)e^{\lambda_2 t}V_2(t) + \cdots + k_n(t)e^{\lambda_n t}V_n(t),$$

where $k_i(t)$'s, $i = 1, 2, \cdots, n$, are periodic functions which are related to initial conditions.

It is well known that there exists a periodic solution if and only if some Floquet exponent has zero real part. For a triangular system, one can have more concrete conclusions about its periodic solutions as stated by the following proposition. We give a different proof instead of considering it as a corollary. By triangular systems, we mean their coefficient matrix $A(t)$ is triangular.

Proposition 2.2 *Let $\dot{X} = A(t)X$, where $A(t)$ is a periodic lower triangular matrix function of dimension $n \times n$. Consider the following n integrals:*

$$\int_0^\omega a_{jj}(t)dt, \quad j = 1, 2, \cdots, n.$$

If there is only one integral $\int_0^\omega a_{ii}(t)dt = 0$, then the system has one family of periodic solutions. If there are k integrals $\int_0^\omega a_{jj}(t)dt = 0$, then the system has at most k families of linearly independent periodic solutions.

Proof Suppose there is only one integral $\int_0^\omega a_{ii}(t)dt = 0$. Set initial values for the first $i-1$ components to be $x_j(0) = 0$ for $j = 1, 2, \cdots, i-1$. It is easy to obtain that $x_j(t) = 0$ for $j = 1, 2, \cdots, i-1$. The ith component is $x_i(t) = \beta e^{\int_0^t a_{jj}(s)ds}$ which is periodic. Now the system is reduced to $n-i$ dimensional nonhomogeneous periodic system with a periodic forcing term

$$\dot{X}_{i+1}(t) = A_{i+1}(t)X_{i+1} + a_{i+1}(t)x_i(t),$$

where $X_{i+1}(t)$ is the column vector obtained from $X(t)$ by deleting the first i components, $A_{i+1}(t)$ is the $(n-i) \times (n-i)$ matrix obtained from $A(t)$ by deleting the first i rows and the first i columns, and $a_{i+1}(t)$ is a column vector obtained from the ith column of $A(t)$ by deleting its first i components. Let $\Phi_{i+1}(t)$ be the fundamental matrix solution of the reduced system $\dot{X}_{i+1}(t) = A_{i+1}(t)X_{i+1}$ with $\Phi_{i+1}(0) = I$. Then $\Phi_{i+1}(t) = P_{i+1}(t)e^{B_{i+1}t}$, where $P_{i+1}(t)$ and B_{i+1} are for the reduced system. By the variation-of-constants formula, any solution can be expressed as:

$$X_{i+1}(t) = \Phi_{i+1}(t)X_{i+1}(0) + \Phi_{i+1}(t)\int_0^t \Phi_{i+1}^{-1}(s)a_{i+1}(s)x_i(s)ds.$$

For a periodic solution, it has to satisfy $X_{i+1}(\omega) = X_{i+1}(0)$. Then we have

$$\int_0^\omega \Phi_{i+1}^{-1}(t)a_{i+1}(t)x_i(t)dt = (e^{-B_{i+1}\omega} - I)X_{i+1}(0).$$

Since there is no zero Floquet for the reduced system, $\det(e^{-B_{i+1}\omega} - I) \neq 0$. There is a unique solution to the initial value $X_{i+1}(0)$ for each $x_i(t) = \beta e^{\int_0^t a_{jj}(s)ds}$. Therefore, there is only one family of periodic solutions parameterized by β.

When there are k integrals $\int_0^\omega a_{jj}(t)dt = 0$, let the fundamental matrix solution be $\Phi(t) = P(t)e^{Bt}$. Changing variable $X(t) = P(t)Y(t)$, the system is transformed to the system $\dot{Y}(t) = BY(t)$. Any periodic solution of this transformed system will result in a periodic solution of the original system. Since B has an eigenvalue with zero real part with the algebraic multiplicity k, the geometric multiplicity of this eigenvalue is equal to or less than k. Then the number of linearly independent periodic solutions to $\dot{Y}(t) = BY(t)$ is at most k. Therefore, the system $\dot{X} = A(t)X$ has at most k families of linearly independent periodic solutions. □

From the proof of Proposition 2.2, one obtains a corollary on the existence of periodic solutions to periodically forced triangular systems.

Corollary 2.6 *Let $\dot{X} = A(t)X + F(t)$, where $A(t)$ is a continuous lower triangular matrix function, $F(t)$ is a continuous vector function, and $A(t+\omega) = A(t)$, $F(t+\omega) = F(t)$. A necessary and sufficient condition in order that for any $F(t)$ the system has periodic solutions of period ω is $\int_0^\omega a_{jj}(t)dt = 0$, $j = 1, 2, \cdots, n$. When the system has periodic solutions, it has a unique periodic solution.*

COROLLARY 2.7 *Let $\dot{X} = A(t)X$, where $A(t)$ is a continuous lower triangular matrix function. If there is some $\int_0^\omega a_{ii}(t)dt > 0$, then the solution $X = 0$ is unstable. If all $\int_0^\omega a_{ii}(t)dt < 0, i = 1, 2, \cdots, n$, then the solution $X = 0$ is uniformly asymptotically stable. If there are $k(\leq n)$ integrals $\int_0^\omega a_{ii}(t)dt = 0$, and all other integrals $\int_0^\omega a_{jj}(t)dt < 0$, then the system is uniformly stable.*

In addition, we briefly discuss the following two types of problems associated with linear periodic systems.

Adjoint systems Given a system $\dot{X} = A(t)X$, its adjoint system is defined by

$$\dot{Y} = -YA(t), \tag{3}$$

where Y is a row vector. It is known that $Y(t) \cdot X(t)$ is a constant when $X(t)$ and $Y(t)$ are solutions of the system $\dot{X} = A(t)X$ and its adjoint system, respectively.[19] When $A(t)$ is a lower triangular matrix function, denote the matrix obtained from $A(t)$ by deleting the last $n - i$ columns and the last $n - i$ rows by $A^i(t)$. By Theorem 3.2 in [19] on the existence of periodic solutions to a nonhomogeneous linear system, we have the following result.

THEOREM 2.6 *Consider the nonhomogeneous triangular linear system $\dot{X} = A(t)X + F(t)$, where $A(t + \omega) = A(t)$ and $F(t + \omega) = F(t)$. Suppose there is only one integer i such that $\int_0^\omega a_{ii}(t)dt = 0$. If $\int_0^\omega Y^i(t)F^i(t)dt = 0$, where $Y^i(t)$ is a nonconstant periodic solution of the adjoint subsystem $\dot{Y}^i = -Y^iA^i(t)$ and $F^i(t)$ is the vector function consisting of the first i components of $F(t)$, then this system has a periodic solution of period ω.*

Proof Since $\int_0^\omega a_{ii}(t)dt = 0$, the homogeneous system $\dot{X} = A(t)X)$ has periodic solutions, and the adjoint system $\dot{Y} = -YA(t)$ also has periodic solutions. The space of periodic solutions is of dimension 1. From the proof of Proposition (2.2), the last $n - i$ components of any periodic solution to the adjoint system are zero, and we only need to consider the subsystem $\dot{Y}^i = -Y^iA^i(t)$. The space of periodic solutions to this subsystem is also of dimension 1. Hence, the orthogonal condition is verified by any nonconstant periodic solution of the subsystem. □

Small periodic perturbations We consider a system of the form $\dot{X} = AX + \varepsilon\phi(t)X$, where ε is a small parameter, A is a constant matrix and X is a vector of dimension >2. It is difficult to analyze such a system in general. However, if $A(t)$ is a triangular periodic matrix, and $\phi(t)$ is also a periodic matrix function, we have a basic stability theorem below.

THEOREM 2.7 *Let $\dot{X} = A(t)X + \varepsilon\phi(t)X$, where $A(t)$ is a continuous lower triangular matrix function, $A(t + \omega) = A(t)$, and $\phi(t)$ is continuous, $\phi(t + \omega) = \phi(t)$, ε is a small parameter. If all integrals $\int_0^\omega a_{ii}(t)dt < 0$, $i = 1, 2, \cdots, n$, then there is $\varepsilon_0 > 0$ such that for all number ε with $|\varepsilon| < \varepsilon_0$, the trivial solution of this system is asymptotically stable. If there is at least one integral $\int_0^\omega a_{jj}(t)dt > 0$, then the trivial solution is unstable.*

Proof Consider the Floquet exponent λ_i of the given system as a single-valued continuous function of the parameter ε, $\lambda_i(\varepsilon)$. Then $\lambda_i(0) = \frac{1}{\omega}\int_0^\omega a_{ii}(t)dt$, where $\omega > 0$ is the smallest positive period. When all $\frac{1}{\omega}\int_0^\omega a_{ii}(t)dt < 0$, by the continuity argument, there exists $\varepsilon_0 > 0$ such that all real parts of the Floquet exponents $R(\lambda_i(\varepsilon)) < 0$ for $|\varepsilon| < \varepsilon_0$. The asymptotic stability follows Theorem 2.2.

Similarly, if some integral $\int_0^\omega a_{jj}(t)dt > 0$, then there is $\varepsilon_0 > 0$ such that the real part of Floquet exponents $R(\lambda_i(\varepsilon)) > 0$ for $|\varepsilon| < \varepsilon_0$. Then the system is unstable. □

2.2. *Nonlinear periodic systems*

Now we conduct some analysis on nonlinear periodic systems. We first look at the trivial equilibrium solution $X = 0$, and discuss the relationship between the stability of the nonlinear system at $X = 0$ and that of its linearized counterpart. By a translation of coordinates, the result can be certainly applied to any constant equilibrium solution.

Consider the system

$$\dot{X} = A(t)X + F(t, X), \tag{4}$$

where $A(t)$ is an $n \times n$ continuous matrix function, and $F(t, X)$ is continuous in t and X and Lipschitz-continuous in X for all $t \in R$ and X in a neighborhood of $X = 0$. Moreover, we assume

$$\lim_{|X| \to 0} \frac{|F(t, X)|}{|X|} = 0 \quad \text{uniformly in } t. \tag{5}$$

Note that the condition in (5) implies that $X = 0$ is a solution to system (4). Then we have the following theorem on the behavior of the trivial solution $X = 0$. This result is an extended version of Theorem 2.4 in [2] or Theorem 7.2 in [16].

Theorem 2.8 *If the trivial solution $X = 0$ of the system $\dot{X} = A(t)X$ is uniformly asymptotically stable for $t \geq 0$, then the trivial solution of* (4) *is also uniformly asymptotically stable. If the trivial solution $X = 0$ of the system $\dot{X} = A(t)X$ is unstable, then the trivial solution of* (4) *is also unstable.*

Next, we discuss the stability of a nonconstant periodic solution to a nonlinear periodic system. Consider

$$\dot{X} = F(t, X), \quad F(t + \omega, X) = F(t, X), \quad t \in R, \quad X \in \Omega, \tag{6}$$

where $F \in C^1(R \times \Omega, R^n)$, Ω is an open and connected subset of R^n. Assuming that a periodic solution to the system (6) exists, we would like to know the stability of the periodic solution. One standard way to achieve this is to linearize the system at the periodic solution. Suppose $p(t) : R \to \Omega$ is a nonconstant periodic solution of the nonlinear system (6) with period ω. One has the variational system, $\dot{Y} = D_X F(t, p(t))Y$, which is a linear periodic system. Then the Floquet exponents of the variational system will be indicators for the stability of the periodic solution $p(t)$ of the nonlinear system (6). In order to use our findings in Floquet theory, we consider a special case; i.e. nonlinear triangular systems. Write $F(t, X) = (F_1(t, X), F_2(t, X), \cdots, F_n(t, X))^T$, and assume $F_1(t, X) = F_1(t, x_1)$, $F_2(t, X) = F_2(t, x_1, x_2), \cdots, F_n(t, X) = F_n(t, x_1, x_2, \cdots, x_n)$. We then study the stability of the periodic solution $p(t)$ by linearizing the system at the solution $p(t)$. Set $Y = X - p(t)$, then $\dot{Y} + \dot{p}(t) = F(t, Y + p(t))$. Since $\dot{p}(t) = F(t, p(t))$, the system becomes $\dot{Y} = F(t, Y + p(t)) - F(t, p(t))$, which can be expanded as

$$\dot{Y} = D_X F(t, p(t))Y + o(|Y|).$$

The variational system with respect to the solution $p(t)$ is

$$\dot{Y} = A(t)Y,$$

where $A(t) = D_X F(t, p(t))$. Our assumptions imply that $A(t)$ is a continuous lower triangular matrix function with period ω, and $\frac{o(|Y|)}{|Y|}$ tends to zero as $Y \to 0$ uniformly in t. Thus, the local stability of the periodic solution $p(t)$ to the nonlinear system (6) can be determined by this linearization. We have the following theorem.

THEOREM 2.9 *If* $\int_0^\omega \frac{\partial F_i}{\partial x_i}(t, p(t))dt < 0$ *for* $i = 1, 2, \cdots, n$, *then the periodic solution* $x = p(t)$ *of the nonlinear system* (6) *is uniformly asymptotically stable; if there is at least one* $\int_0^\omega \frac{\partial F_j}{\partial x_j}(t, p(t))dt > 0$, *then the periodic solution* $x = p(t)$ *is unstable.*

Proof By Theorem 4.2.1 in [20], the periodic solution is uniformly asymptotically stable when every Floquet exponent has negative real part. By Theorem 2.5, when all the integrals $\int_0^\omega \frac{\partial F_i}{\partial x_i}(t, p(t))dt < 0$ for $i = 1, 2, \cdots, n$, each Floquet exponent has negative real part. Hence, the first part of the Theorem is verified. When some integral $\int_0^\omega \frac{\partial F_j}{\partial x_j}(t, p(t))dt > 0$, the corresponding Floquet exponent has positive real part. By Theorem 4.2.1 in [20], the periodic solution is unstable. □

In particular, when the nonlinear system (6) is autonomous,

$$\dot{X} = F(X), \tag{7}$$

the analysis can be simplified and the stability of periodic solutions can be investigated by the stability of the fixed points of Poincaré map. Let $X = p(t)$ be a periodic solution to the system (7) with $X_0 = p(0)$ and period ω, and $P_o(X)$ be the Poincaré map at X_0, where X_0 is a fixed point of the Poincaré map that corresponds to the periodic solution $X = p(t)$. The local stability of the periodic solution $X = p(t)$ will be determined by the eigenvalues of the derivative of the Poincaré map $P_o(X)$ at X_0. Specifically, let the derivative of $F(X)$ along the periodic solution $p(t)$ be $DF(p(t))$. Then $DF(p(t))$ is a periodic function of t. Consider the linear system

$$\dot{X} = DF(p(t))X. \tag{8}$$

By the Floquet Theorem 2.1, the fundamental matrix solution is $\Phi(t) = P(t)e^{Bt}$. There is one Floquet exponent that is zero since $X = \dot{p}(t)$ is a periodic solution of the linear system (8). Define the matrix

$$H(t, X) = D\phi_t(X),$$

which satisfies

$$\frac{\partial H(t, X)}{\partial t} = DF(\phi_t(X))H(t, X),$$

where $\phi_t(X) = \phi(t, X)$ is the flow of the system (7) and $p(t) = \phi_t(X_0)$. When $X = X_0$, the function $\Phi(t) = H(t, X_0)$ satisfies

$$\dot{\Phi}(t) = DF(p(t))\Phi(t),$$

and $\Phi(0) = I$. Therefore,

$$H(t, X_0) = P(t)e^{Bt},$$

and

$$H(\omega, X_0) = e^{B\omega}.$$

On the other hand, the derivative of the Poincaré map $DP_o(X_0)$ is a $(n-1)\times(n-1)$ matrix obtained from $H(\omega, X_0)$ by deleting one column and one row which correspond to the eigenvalue 1. For more details, we refer to Theorem 2 on page 223 in [21]. When the system (7) is triangular, namely $F(X) = \big(F_1(x_1), F_2(x_1, x_2), \cdots, F_n(x_1, x_2, \cdots x_n)\big)^T$, it is clear to see that $DF(p(t))$ is a lower triangular matrix function. By Theorem 2.5, we can easily compute the eigenvalues of the derivative of the Poincaré map $P_o(X)$ at X_0. They are given in the following corollary. Using this information, we can easily construct stable, unstable and center manifolds for periodic orbits of autonomous triangular systems.

Corollary 2.8 *Suppose the system (7) is triangular, which has a periodic solution $X = p(t)$ with $X_0 = p(0)$ and period ω. Let $P_o(X)$ be the Poincaré map at X_0. Then, the characteristic exponents of $p(t)$ are $\lambda_j = \frac{1}{\omega}\int_0^{\omega}\frac{\partial F_j}{\partial x_j}(p(t)dt$, $j = 1, 2, \cdots, n$, and there is at least one zero characteristic exponent; the eigenvalues of the derivative of the Poincaré map $P_o(X)$ are $e^{\lambda_j\omega}$, where $j = 1, 2, \cdots, n$ except for one index corresponding to the zero characteristic exponent.*

2.3. *Delay linear periodic systems*

We now extend our analysis to systems of delay differential equations. There is a theory of delay linear periodic systems that is analogous to the Floquet theory for linear ordinary differential equations. One interesting question one may ask is how the Floquet exponents for both delay and nondelay linear systems are related when both share the same periodic coefficients. We will explore this question when the coefficient matrices are periodic triangular matrices. Before we state our results, we briefly introduce the Floquet theory for delay linear systems; for details, we refer to [19,22,23].

Let $C = C[-\tau, 0]$ for some $\tau > 0$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n. Suppose $L : R\times C \longrightarrow R^n$ is a linear functional for any fixed $t \in R$, and there is $\omega > 0$ such that $L(t+\omega, \phi) = L(t, \phi)$ for all t and $\phi \in C$. Let $X_t \in C$ defined by $X_t(\theta) = X(t+\theta)$ for $\theta \in [-\tau, 0]$ and $t \in R$. The delay linear system is given by

$$\dot{X}(t) = L(t, X_t). \tag{9}$$

Using Riesz linear functional representation theorem, this equation can be expressed as a Lebesgue–Stieltjes integral of a bounded variation matrix function. We will focus on some special forms of this matrix. Define the operator U by $U\phi = X_\omega(0, \phi)$ where $X(t, \phi)$ is the solution to the system (9). It is easy to check that U is a compact operator. The compact operator U induces a decomposition of C, and in some invariant subspaces, the solutions have Floquet representations. The following facts follow from the compact operator U. The spectrum $\sigma(U)$ is at most countable and is a compact set in the complex plane. $0 \in \sigma(U)$ since C is of infinite dimension and 0 is the only limit point in $\sigma(U)$ when $\sigma(U)$ is infinite. For $\rho \in \sigma(U)$ and $\rho \neq 0$, there are closed invariant subspaces $K(\rho)$ and $E(\rho)$ and $\dim E < \infty$, such that $C = K(\rho) \oplus E(\rho)$. When U is restricted to the subspace $E(\rho)$, U can be represented by a matrix M under a given basis of $E(\rho)$. The matrix M is nonsingular and has a unique eigenvalue ρ. So, define $B = \frac{1}{\omega}\ln M$, and $p(t) = X(t, \phi)e^{-Bt}$, where $X(t, \phi) \in E(\rho)$. It is easy to check that $X(t, \phi) = p(t)e^{-Bt}$ is a solution, where $p(t)$ is periodic. Then ρ is a characteristic multiplier, and $\ln \rho$ is a Floquet exponent.

Let $\rho_1, \rho_2, \cdots, \rho_n, \cdots$, be the characteristic multipliers of the system (9) ordered in such a way that $|\rho_n| \geq |\rho_{n+1}|$. Let $A_n = \{\rho_1, \rho_2, \cdots, \rho_n\}$ and $R_n = \{\rho_{n+1}, \cdots\}$ where $|\rho_{n+1}| < \alpha < 1$. From the property of compact operator, this exists only a finite number of multipliers outside the circle $|z| \leq \alpha$. For the decomposition $\sigma(U) = A_n \cup R_n \cup 0$, there is a direct decomposition $C = E(\rho_1) \oplus E(\rho_2) \oplus \cdots \oplus E(\rho_n) \oplus C_0$, and $\sigma(U|C_0) = R_n \cup 0$. If $\phi \in C$, then $\phi = \phi_n + \varphi$ and $X(t, \phi) = X(t, \phi_n) + X(t, \varphi)$. The solution $X(t, \phi_n)$ is a linear combination of Floquet-type solutions, and the solution $X(t, \varphi)$ approaches zero as $e^{-\beta t}$, where $\beta = -\frac{1}{\omega} \ln \alpha$.

Now we consider the case when the delay is specifically given. For this general case, consider the system

$$\dot{X}(t) = \sum_{i=0}^{m} B_i(t) X(t - k_i \omega), \quad B_i(t + \omega) = B_i(t), \tag{10}$$

where $B_i(t)$ is $n \times n$ continuous matrix function, and $k_i \geq 0$ is an integer. For the exact calculation, we will focus on special cases $\dot{X}(t) = B(t)X(t - \omega)$ and $\dot{X}(t) = A(t)X(t) + B(t)X(t - \omega)$. When $A(t)$ and $B(t)$ both are triangular matrix functions, for the nondelay linear systems $\dot{X}(t) = B(t)X(t)$ and $\dot{X}(t) = A(t)X(t)$, we know how to calculate their Floquet exponents exactly. For the delay linear system, we can describe the relations between the Floquet exponents of the nondelay systems and that of corresponding delay systems. For nontriangular periodic matrices, we do not have an answer yet.

THEOREM 2.10 *Let $\dot{X} = B(t)X(t - \omega)$, where $B(t)$ is a continuous lower triangular matrix function, $B(t + \omega) = B(t)$. Let $\lambda_i = \frac{1}{\omega} \int_0^\omega b_{ii}(t) dt$ be the Floquet exponents of the system $\dot{X} = B(t)X(t)$ for $i = 1, 2, \cdots, n$, and μ be the Floquet exponent of the system $\dot{X} = B(t)X(t - \omega)$. Then $\lambda_i = \mu e^{\mu\omega}$; $Re(\mu) < 0$ if and only if $-\frac{\pi}{2\omega} < \lambda_i < 0$ for $i = 1, 2, \cdots, n$. Consequently, the solution $X = 0$ to the delay system is uniformly asymptotically stable if and only if $-\frac{\pi}{2\omega} < \lambda_i < 0$, $i = 1, 2, \cdots, n$.*

Proof From the definition of characteristic multipliers of delay linear differential equations, $\rho = e^{\mu\omega}$ is a characteristic multiplier of the linear periodic system

$$\dot{X} = B(t)X(t - \omega) \tag{11}$$

if and only if there is a nonzero n-vector function $v(t) = v(t + \omega)$ such that $X = v(t)e^{\mu t}$ satisfies the system (11). Hence, one has a linear system

$$\dot{v}(t) = (-\mu I + B(t)e^{-\mu\omega})v(t). \tag{12}$$

As in the proof of Theorem (2.5), there is a fundamental matrix solution $V(t, \lambda)$ of the system (12), which has diagonal entries, $e^{\int_0^t (b_{ii}(s)e^{-\mu\omega} - \mu)ds}$, $i = 1, 2, \cdots, n$, and $V(0, \lambda) = I$. Then $v(t) = V(t, \mu)v(0)$, and the initial value $v(0) \neq 0$ must be chosen such that $v(t) = v(t + \omega)$. Such a $v(0) \neq 0$ exists if and only if μ satisfies the characteristic equation $\det(V(\omega, \mu) - I) = 0$. Therefore

$$e^{\int_0^\omega (b_{ii}(s)e^{-\mu\omega} - \mu)ds} - 1 = 0, \quad i = 1, 2, \cdots, n.$$

Therefore, we have $\lambda_j e^{-\mu\omega} - \mu = 0$, or $\lambda_j = \mu e^{\mu\omega}$, where $\lambda_j = \frac{1}{\omega}\int_0^\omega b_{jj}(t)dt$ is the Floquet exponent of the nondelay system $\dot{X} = B(t)X(t)$, $j = 1, 2, \cdots, n$. It is known for each $\lambda_i \neq 0$, there are at most countably infinitely many μ that satisfy $\lambda_j = \mu e^{\mu\omega}$.

Let $z = \mu\omega$. We have $ze^z - \omega\lambda_j = 0$. The Floquet exponent μ of the delay system (11) has negative real part if and only if z roots of $ze^z - \omega\lambda_j = 0$ has negative real parts since $\omega > 0$ is the period. By Theorem A.5 in [22], all roots of the equation $ze^z - \omega\lambda_j = 0$ have negative real parts if and only if $-\omega\lambda_j > 0$ and $-\omega\lambda_j < \frac{\pi}{2}$. That is, $-\frac{\pi}{2\omega} < \lambda_j < 0$. Therefore, in order for all Floquet exponents μ to have negative real parts, all λ_i have to satisfy $-\frac{\pi}{2\omega} < \lambda_i < 0$ for $i = 1, 2, \cdots, n$. □

COROLLARY 2.9 *Let $\dot{X} = B(t)X(t-\omega)$, where $B(t)$ is a lower triangular continuous matrix function, and $B(t+\omega) = B(t)$. If $\int_0^\omega b_{ii}(t)dt > 0$ or $\int_0^\omega b_{ii}(t)dt < -\frac{\pi}{2}$, then the solution $X = 0$ is unstable.*

THEOREM 2.11 *Let $\dot{X} = A(t)X(t) + B(t)X(t-\omega)$, where $A(t)$ and $B(t)$ are continuous lower triangular matrix functions, $A(t+\omega) = A(t)$, and $B(t+\omega) = B(t)$. Let $\lambda_i(A)$ and $\lambda_i(B)$ be the Floquet exponents of the systems $\dot{X} = A(t)X(t)$ and $\dot{X} = B(t)X(t)$ respectively, and μ be the Floquet exponent of the given delay system. Then,*

$$\lambda_i(A) + \lambda_i(B)e^{-\mu\omega} - \mu = 0;$$

$Re(\mu) < 0$ if and only if

$$\lambda_i(A) < \frac{1}{\omega}, \quad -\lambda_i(A)\sec\xi_i < \lambda_i(B) < -\lambda_i(A),$$

where ξ_i is the root of $\xi_i = \lambda_i(A)\omega\tan\xi_i$, $0 < \xi_i < \pi$, $i = 1, 2, \cdots, n$. Consequently, the solution $X = 0$ is uniformly asymptotically stable if and only if $\int_0^\omega a_{ii}(t)dt < 1$, $-\frac{1}{\cos\xi_i}\int_0^\omega a_{ii}(t)dt < \int_0^\omega b_{ii}(t)dt < -\int_0^\omega a_{ii}(t)dt$, $i = 1, 2, \cdots, n$.

Proof As in the proof of Theorem 2.10, let $\rho = e^{\mu\omega}$ be a characteristic multiplier. Then there is a nonzero n-vector function $v(t) = v(t+\omega)$ such that $X = v(t)e^{\mu t}$ satisfies the given delay system. Hence, we have $\dot{v}(t) = (-\mu I + A(t) + B(t)e^{-\mu\omega})v(t)$. As in the proof of Theorem 2.5, there is a fundamental matrix solution $V(t,\mu)$, which has diagonal entries, $e^{\int_0^t (a_{jj}(s)+b_{jj}(s)e^{-\mu\omega}-\mu)ds}$, $j = 1, 2, \cdots, n$, and $V(0,\mu) = I$. Then $v(t) = V(t,\mu)v(0)$, and the initial value $v(0) \neq 0$ must be chosen such that $v(t) = v(t+\mu)$. Such a $v(0) \neq 0$ exists if and only if μ satisfies the characteristic equation $\det(V(\omega,\mu) - I) = 0$. Therefore

$$e^{\int_0^\omega (a_{jj}(s)+b_{jj}(s)e^{-\mu\omega}-\mu)ds} - 1 = 0, \qquad j = 1, 2, \cdots, n.$$

Therefore, we have $\lambda_j(A) + \lambda_j(B)e^{-\mu\omega} - \mu = 0$, where $\lambda_j(A) = \frac{1}{\omega}\int_0^\omega a_{jj}(t)dt$, $\lambda_j(B) = \frac{1}{\omega}\int_0^\omega b_{jj}(t)dt$ are the Floquet exponents of the systems $\dot{X} = A(t)X(t)$ and $\dot{X} = B(t)X(t)$ respectively, $j = 1, 2, \cdots, n$.

Let $z = \mu\omega$. We have $ze^z - \lambda_j(A)\omega e^z - \lambda_j(B)\omega = 0$. The Floquet exponent μ of the delay system has negative real part if and only if z roots of $ze^z - \lambda_j(A)\omega e^z - \lambda_j(B)\omega = 0$ have negative real parts since $\omega > 0$ is the period. By Theorem A.5 in [22], all roots of the equation $ze^z - \lambda_j(A)\omega e^z - \lambda_j(B)\omega = 0$ have negative real parts if and only if

$$-\lambda_j(A)\omega > -1, \quad -\lambda_j(A)\omega - \lambda_j(B)\omega > 0,$$
$$-\lambda_j(B)\omega < \xi_j \sin\xi_j + \lambda_j(B)\omega\cos\xi_j,$$

where ξ_j is the root of $\xi_j = \lambda_j(B)\omega\tan\xi_j$. Since $\xi_j = \lambda_j(A)\omega\tan\xi_j = \xi_j = \lambda_j(A)\omega\frac{\sin\xi_j}{\cos\xi_j}$, one has $\xi_j\sin\xi_j + \lambda_j(A)\omega\cos\xi_j = \lambda_j(A)\omega\frac{\sin^2\xi_j}{\cos\xi_j} + \lambda_j(A)\omega\cos\xi_j = \lambda_j(A)\omega\frac{1}{\cos\xi_j}$, and $-\lambda_j(A)\omega\frac{1}{\cos\xi_j} < \lambda_j(B)\omega$. Combining the second condition, one has $-\lambda_j(A)\frac{1}{\cos\xi_j} < \lambda_j(B) < -\lambda_j(A)$. Therefore, the conditions can be expressed as $\int_0^\omega a_{jj}(t)dt < 1$, $-\frac{1}{\cos\xi_j}\int_0^\omega a_{jj}(t)dt < \bar{b}_j < -\int_0^\omega a_{jj}(t)dt$, where ξ_j is the root of $\xi_j = \lambda_j(A)\omega\tan\xi_j$, $0 < \xi_j < \pi$, $j = 1, 2, \cdots, n$. □

Using similar arguments, we can obtain the result below.

THEOREM 2.12 *Given $\dot{X}(t) = \sum_{j=0}^{m} B_j(t)X(t - k_j\omega)$, where $B_j(t+\omega) = B_j(t)$ is lower triangular and $k_j \geq 0$ is an integer for $j = 0, 1, \cdots, m$. Let $\lambda_i(B_j)$ be a Floquet exponent of the system $\dot{X}(t) = B_jX(t)$ for $i = 1, 2, \cdots, n$ and $j = 0, 1, \cdots, m$; also let μ be a Floquet exponent of the given delay system. Then*

$$\sum_{j=0}^{m}\lambda_i(B_j)e^{-k_j\omega\mu} - \mu = 0; \qquad i = 1, 2, \cdots, n. \tag{13}$$

3. An application to cholera modeling

In this section, we propose and analyze a mathematical model for cholera dynamics with seasonal oscillation, using our findings in Floquet theory. Particularly, we will determine threshold values of cholera epidemics in various periodic environments.

Cholera is a severe intestinal infection caused by the bacterium *Vibrio cholerae*. Many epidemic models have been published [24–28] in recent years to investigate cholera dynamics. Among these Codeço [24] was the first to explicitly incorporate bacterial dynamics into a SIR epidemiological model. Meanwhile, cholera is a seasonal disease in many endemic places and infection peaks often occur annually in the rainy or monsoon season.[11,13] Currently, very few mathematical cholera models have taken into account the seasonal factors, partly due to the challenges in model analysis and validation. In addition, several recent studies [12,29,30] have shown that lytic bacteriophage specific for *Vibrio cholerae* can suppress the growth of the bacteria and thus reduce the severity of cholera outbreak.

To better understand the complex interaction among human population, vibrio, phage, and seasonality in cholera dynamics, we construct a new model as follows. This is a significant extension of Codeço's model [24] by incorporating the phage dynamics and the seasonal oscillation of cholera transmission. Our model is represented by

$$\frac{dS}{dt} = n(H - S) - a\frac{B}{K+B}S,$$
$$\frac{dI}{dt} = a\frac{B}{K+B}S - rI,$$
$$\frac{dB}{dt} = eI - mB - \delta\frac{B}{\widetilde{K}+B}P,$$

$$\frac{dP}{dt} = \xi I + \kappa \frac{B}{\widetilde{K} + B} P - \nu P, \tag{14}$$

where S is the susceptible human population, I is the infectious human population, and B and P are the concentrations of the pathogen (i.e. vibrio) and the phage, respectively, in the contaminated water. The total human population, H, is assumed to be a constant. The parameter n denotes the natural human birth/death rate, a denotes the human contact rate to the vibrio, δ is the death rate of the bacteria due to phage predation, κ is the growth rate of the phage due to feeding on the vibrio, e and ξ are the rates of human contribution (e.g. by shedding) to the pathogen and the phage, respectively, and m and ν are the natural death rates of the vibrio and the phage, respectively. In addition, $r = n + \gamma$ with γ being the recovery rate, and K and $\widetilde{K}$ are the half saturation rates of the vibrio in the interaction with human and phage, respectively.

To investigate the impact of seasonality on cholera dynamics, we will particularly examine the periodic variation of three parameters, m, e and a, and apply the results from Floquet theory in the analysis that follows.

Let us first assume that the parameter m is a positive periodic function of time, $m(t)$, representing a seasonal variation of the extinction rate of the vibrio.

It is clear to see that $E_0 = (H, 0, 0, 0)$ is the unique *disease free equilibrium* (DFE) of the system. For ease of discussion, we translate the DFE to the origin via a change of variable by $\bar{S} = H - S$. Then, with a linearization at (0, 0, 0, 0), the original system becomes

$$\begin{aligned}
\frac{d\bar{S}}{dt} &= -n\bar{S} + \frac{aH}{K}B + \left(\frac{aB(H-\bar{S})}{K+B} - \frac{aH}{K}B\right),\\
\frac{dI}{dt} &= -rI + \frac{aH}{K}B + \left(\frac{aB(H-\bar{S})}{K+B} - \frac{aH}{K}B\right),\\
\frac{dB}{dt} &= eI - m(t)B - \delta\frac{B}{\widetilde{K}+B}P,\\
\frac{dP}{dt} &= \xi I - \nu P + \kappa\frac{B}{\widetilde{K}+B}P.
\end{aligned} \tag{15}$$

Thus, system (15) can be written in a compact form

$$\dot{X} = A(t)X + F(t, X), \tag{16}$$

with $X = \left(\bar{S}, I, B, P\right)^T$ and the matrix

$$A(t) = \begin{pmatrix} -n & 0 & \frac{aH}{K} & 0 \\ 0 & -r & \frac{aH}{K} & 0 \\ 0 & e & -m(t) & 0 \\ 0 & \xi & 0 & -\nu \end{pmatrix}. \tag{17}$$

In order to study the trivial solution of (16), we first check the condition (5). We have

$$\begin{aligned}
|F(t,X)| &\le 2\Big|\frac{aH}{K}B - \frac{aB(H-\bar{S})}{K+B}\Big| + (\delta+\kappa)\frac{|BP|}{\widetilde{K}}\\
&= 2\Big|\frac{aH}{K}B - \frac{aB(H-\bar{S}))}{K}\Big(1 - \frac{B}{K} + \frac{B^2}{K^2} - \cdots\Big)\Big| + (\delta+\kappa)\frac{|BP|}{\widetilde{K}}\\
&= 2\Big| - \frac{aB\bar{S}}{K} + \frac{a(H-\bar{S})B^2}{K^2} - \frac{a(H-\bar{S})B^3}{K^3} + \cdots\Big| + (\delta+\kappa)\frac{|BP|}{\widetilde{K}}\\
&\le 2\frac{a|B\bar{S}|}{K} + 2\frac{a|H-\bar{S}|B^2}{K^2} + 2\frac{a|(H-\bar{S})B^3|}{K^3} + (\delta+\kappa)\frac{|BP|}{\widetilde{K}}.
\end{aligned}$$

Then, it is straightforward to check that $\lim_{|X|\to 0}\frac{|F(t,X)|}{|X|} = 0$ uniformly in t. Based on Theorem 2.8, we only need to consider the periodic linear system

$$\dot{X} = A(t)X, \tag{18}$$

where the matrix $A(t)$ is given in Equation (17).

It is easy to observe that the matrix $A(t)$ has a block tridiagonal structure. From Theorem 2.5 and Corollary 2.5, two Floquet exponents of the system (18) are given by $-n$ and $-\nu$; the other two Floquet exponents are determined by the matrix block $\begin{pmatrix} -r & aH/K \\ e & -m(t) \end{pmatrix}$. Hence, its stability depends on the 2×2 sub-system

$$\dot{y} = \begin{pmatrix} -r & \frac{aH}{K} \\ e & -m(t) \end{pmatrix} y. \tag{19}$$

Let

$$m_1 = \min_{0\le t\le\omega} m(t) \text{ and } M = \max_{0\le t\le\omega} m(t).$$

We have the following lemma which describes the stability of the sub-system (19).

Lemma 3.1 *When $m_1 > \frac{aeH}{rK}$, the trivial solution of* (19) *is asymptotically stable; when $M < \frac{aeH}{rK}$, the trivial solution is unstable; when $m_1 \le \frac{aeH}{rK} \le M$, the system* (19) *may have a periodic solution.*

Proof By Theorem 2.4, the system (19) has two distinct real characteristic multipliers and two distinct real Floquet exponents λ_1 and λ_2. So, its solution takes the form of $e^{\lambda t}p(t) = e^{\lambda t}\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$, where $p(t+\omega) = p(t)$ and λ is λ_1 or λ_2. Denoting $b = \frac{aH}{K}$, and substituting this solution into the system, we obtain

$$\begin{aligned}
\dot{p}_1(t) &= -(\lambda + r)p_1(t) + bp_2(t),\\
\dot{p}_2(t) &= ep_1(t) - (\lambda + m(t))p_2(t).
\end{aligned}$$

Then, integrating both sides of these two equations and noting that $p_i(0) = p_i(\omega)$, $i = 1, 2$, we have

$$0 = -(\lambda + r)\int_0^\omega p_1(t)dt + b\int_0^\omega p_2(t)dt, \tag{20}$$

$$0 = e\int_0^\omega p_1(t)dt - \int_0^\omega (\lambda + m(t))p_2(t)dt. \tag{21}$$

Multiplying Equation (20) by e and Equation (21) by $(\lambda + r)$, and adding the results, we obtain

$$0 = \int_0^{\omega} [eb - (\lambda + r)(\lambda + m(t))] p_2(t) dt.$$

Since $p_2(t) > 0$ (as can be seen from the proof of Theorem 2.4), the equation $eb - (\lambda + r)(\lambda + m(t)) = 0$, or

$$eb = (\lambda + r)(\lambda + m(t)), \tag{22}$$

must have a solution on $[0, \omega]$. We now consider all three different cases.

When $m_1 > \frac{eb}{r}$, we consider $\lambda + m(t) = \frac{eb}{\lambda + r}$ which is equivalent to Equation (22). If $\lambda = 0$, then $m(t) = \frac{eb}{r}$. Since $m(t) \geq m_1 > \frac{eb}{r}$, it is impossible that Equation (22) has a solution. If $\lambda > 0$, then $\frac{eb}{\lambda + r} < \frac{eb}{r}$, and $m_1 > \lambda + m(t)$. This is a contradiction. Therefore, both λ_1 and λ_2 must be negative, and the trivial solution is asymptotically stable. Similarly, it can be shown that when $M < \frac{eb}{r}$, one of the two characteristic multipliers must be positive, so that the trivial solution is unstable.

Now we consider the third case. When $m_1 \leq \frac{eb}{r} \leq M$, we have $m_1 r \leq eb \leq Mr$. Suppose that Equation (22) has a root t_0; that is, $eb = (\lambda + r)(\lambda + m(t_0))$. Regrouping this equation, we have $\lambda^2 + (r + m(t_0))\lambda + rm(t_0) - eb = 0$. This is a quadratic equation so that $\lambda_1 \lambda_2 = rm(t_0) - eb$. Combining with the assumption on $\frac{eb}{r}$, we obtain

$$r(m(t_0) - M) \leq rm(t_0) - eb \leq r(m(t_0) - m_1).$$

Since $r(m(t_0) - M) \leq 0$ and $r(m(t_0) - m_1) \geq 0$, it is possible that $rm(t_0) - eb = 0$. When $m(t_0) = \frac{eb}{r}$, one Floquet is zero and there is a periodic solution. □

Summarizing our analysis above for periodic $m(t)$, we can now state the results below.

THEOREM 3.1 *Assume that system (14) has a positive periodic $m(t)$. When $m_1 > \frac{aeH}{rK}$, the disease free equilibrium $E_0 = (H, 0, 0, 0)$ is uniformly asymptotically stable. When $M < \frac{aeH}{rK}$, the disease free equilibrium is unstable.*

We can now conduct similar study on the second scenario, by setting the parameter e as a positive periodic function $e(t)$, representing a seasonal oscillation of the per capita contamination rate, i.e. the unit rate of human contribution (e.g. shedding) to the pathogen in the environment.

Following similar arguments as before, we obtain that the stability of this system depends on the 2×2 sub-system

$$\dot{y} = \begin{pmatrix} -r & \frac{aH}{K} \\ e(t) & -m \end{pmatrix} y. \tag{23}$$

Let

$$e_1 = \min_{0 \leq t \leq \omega} e(t) \text{ and } E = \max_{0 \leq t \leq \omega} e(t).$$

We then have the following lemma which describes the stability of the sub-system (23).

LEMMA 3.2 *When $e_1 > \frac{mrK}{aH}$, the trivial solution of system (23) is unstable; when $E < \frac{mrK}{aH}$, the trivial solution is asymptotically stable; when $e_1 \leq \frac{mrK}{aH} \leq E$, the system (23) may have a periodic solution.*

The proof of Lemma 3.2 is similar to that of Lemma (3.1) and is omitted here. As a result, we obtain:

THEOREM 3.2 *Assume that system (14) has a positive periodic $e(t)$. When $E < \frac{mrK}{aH}$, the disease free equilibrium $E_0 = (H, 0, 0, 0)$ is uniformly asymptotically stable. When $e_1 > \frac{mrK}{aH}$, the disease free equilibrium is unstable.*

In addition, we can consider the third scenario by setting the parameter a as a positive periodic function $a(t)$, representing a seasonal variation of the contact rate. The stability now depends on the sub-system

$$\dot{y} = \begin{pmatrix} -r & \frac{a(t)H}{K} \\ e & -m \end{pmatrix} y. \tag{24}$$

Let

$$a_1 = \min_{0 \le t \le \omega} a(t) \qquad \text{and} \qquad A = \max_{0 \le t \le \omega} a(t).$$

We then have the following results which describe the stability property in this case.

LEMMA 3.3 *When $a_1 > \frac{mrK}{eH}$, the trivial solution of (24) is unstable; when $A < \frac{mrK}{eH}$, the trivial solution is asymptotically stable; when $a_1 \le \frac{mrK}{eH} \le A$, the system (24) may have a periodic solution.*

THEOREM 3.3 *Assume that system (14) has a positive periodic $a(t)$. When $A < \frac{mrK}{eH}$, the disease free equilibrium $E_0 = (H, 0, 0, 0)$ of the system is uniformly asymptotically stable. When $a_1 > \frac{mrK}{eH}$, the disease free equilibrium is unstable.*

In principle, we can also vary other parameters in the cholera model as time periodic and analyze the periodic systems in a similar way. An interesting observation is that phage dynamics do not change the stability of the disease free equilibrium. Instead, the growth or decay of phage density in the environment will affect the infection size in both short- and long-term dynamics. Below we will present numerical simulation results for demonstration. Additionally, if we know a nontrivial periodic solution to our cholera model (14), we can investigate the stability of the periodic solution as well by using the theory presented in the previous section. It is reasonable to expect more specific conditions for the existence of nonconstant periodic solutions to our cholera model in various periodic environments, and we plan to explore these in a future study. Our numerical results will be able to provide useful insight.

The base values of the parameters in our cholera model (14) are taken from [24,30]. In particular, $H = 10{,}000$, $n = 0.001\,(\text{day})^{-1}$, $a = 1\,(\text{day})^{-1}$, $K = 10^6$ cells/ml, $r = 0.2\,(\text{day})^{-1}$, $m = 0.33\,(\text{day})^{-1}$, $e = 10$ cells/(ml · day).

For the first scenario, we replace m by a periodic function

$$m(t) = c_1 + c_2 \sin(2\pi t/365)$$

with a period $\omega = 365$ days, representing annual variation of the bacterial extinction rate. Based on Theorem 3.1, the stability threshold value is

$$\frac{aeH}{rK} = 0.5.$$

We then conduct numerical simulation to this model with various choices of the constants c_1 and c_2 and differing initial conditions, and the simulation results show consistent patterns. A typical set of numerical results are presented here. Figure 1(a) shows the infection curve (I) versus time (t) with $c_1 = 0.7$ and $c_2 = 0.19$. The initial infection is set as $I(0) = 100$. In this case, $m_1 = \min m(t) = 0.51 > 0.5$, and we observe the number of the infectious individuals quickly drops to zero and stays there forever; i.e. the disease dies out. This is an evidence that the disease free equilibrium is uniformly asymptotically stable. In contrast, Figure 1(b) shows the result with $c_1 = 0.25$ and $c_2 = 0.24$, and with the initial condition $I(0) = 1$. In this case, $M = \max m(t) = 0.49 < 0.5$, and we expect an unstable disease free equilibrium. Indeed we observe that even though only one infectious individual is introduced initially, an epidemic outbreak quickly develops with a peak value at about 1200, or 12% of the total population ($H = 10{,}000$). Afterwards, the disease persists and epidemics occur in every few years in a periodic manner with peak values of the outbreaks at about 15% of the total population, implying that seasonal oscillation of the bacterial extinction rate has large impact on disease epidemicity and leads to high infection levels from time to time. Figure 1(c) and (d) show the time series for the bacterial density (B) and the phage density (P) in this case, respectively, with a similar pattern. For comparison, we now remove the phage (P) from the model (14) and run the simulation under the same parameter settings; particularly, $c_1 = 0.25$, $c_2 = 0.24$, and $I(0) = 1$. Results are displayed in Figure 2 for I vs. t and B vs. t. By comparing Figures 2(b) and (c), we clearly see that

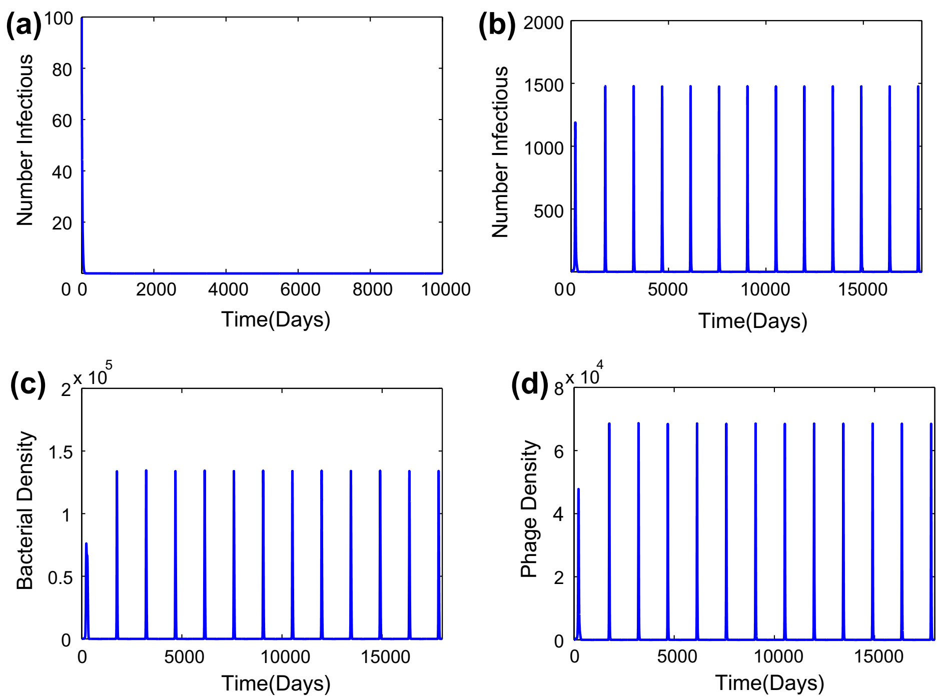

Figure 1. Time series for model (14) with periodic bacterial extinction rate $m(t) = c_1 + c_2 \sin(2\pi t/365)$. (a) $c_1 = 0.7$, $c_2 = 0.19$. The disease free equilibrium is uniformly asymptotically stable. (bcd) $c_1 = 0.25$, $c_2 = 0.24$. The disease free equilibrium is unstable and disease persists.

the peak values for the bacterial concentration without phage are significantly higher than those with page. Correspondingly, by looking at Figures 2(a) and 1(b), we also observe that the peak infections in the case without phage are always higher than those with phage. This observation is consistent with the findings in [12,29,30] that vibriophage can reduce the severity of cholera outbreak.

For the second scenario, we replace the base value of e by a periodic function

$$e(t) = c_1 + c_2 \sin(2\pi t/365)$$

with a period $\omega = 365$ days, representing annual variation of the human contribution rate to the vibrio in the contaminated water. Based on Theorem 3.2, the stability threshold value is

$$\frac{mrK}{aH} = 6.6.$$

Two typical simulation results are presented in Figure 3. We first set $c_1 = 4.0$ and $c_2 = 2.5$, so that $E = \max\ e(t) = 6.5 < 6.6$. Figure 3(a) clearly shows the disease dies out in this case, where the initial condition is $I(0) = 100$. Next we set $c_1 = 8.2$ and $c_2 = 1.5$, so that $e_1 = \min\ e(t) = 6.7 > 6.6$. Figure 3(b) shows that a high epidemic level occurs shortly

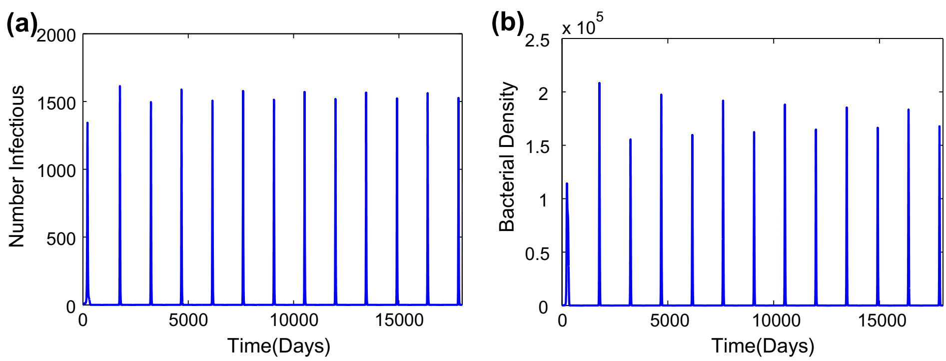

Figure 2. Time series for model (14) without phage; $m(t) = c_1 + c_2 \sin(2\pi t/365)$ where $c_1 = 0.25$, $c_2 = 0.24$.

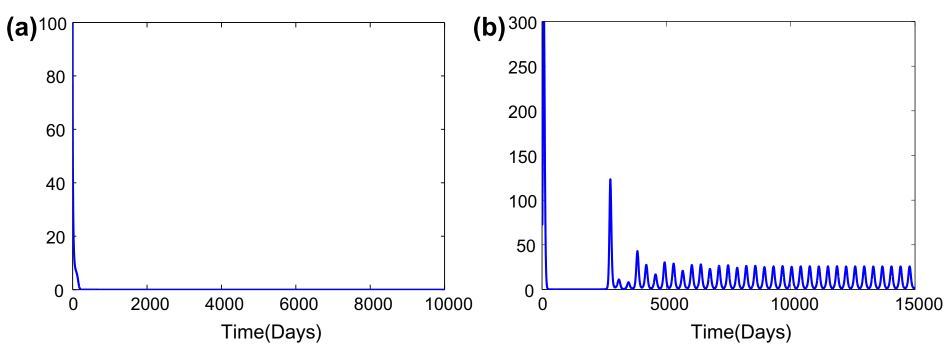

Figure 3. Time series for model (14) with periodic human contribution rate to the pathogen $e(t) = c_1 + c_2 \sin(2\pi t/365)$. (a) $c_1 = 4.0$, $c_2 = 2.5$. The disease free equilibrium is uniformly asymptotically stable. (b) $c_1 = 8.2$, $c_2 = 1.5$. The disease free equilibrium is unstable.

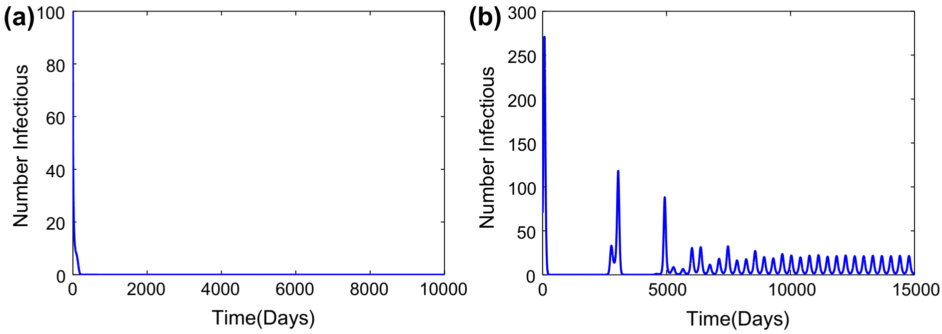

Figure 4. Time series for model (14) with periodic exposure rate $a(t) = c_1 + c_2 \sin(2\pi t/365)$. (a) $c_1 = 0.4, c_2 = 0.25$. The disease free equilibrium is uniformly asymptotically stable. (b) $c_1 = 0.8$, $c_2 = 0.13$. The disease free equilibrium is unstable.

after the initial time, followed by persistent epidemic oscillations over time with low peak values. This represents an unstable disease free equilibrium.

Finally, for the third scenario, we replace the base value of a by a periodic function

$$a(t) = c_1 + c_2 \sin(2\pi t/365)$$

with a period $\omega = 365$ days, representing annual oscillation of the rate of exposure to the contaminated water. Based on Theorem 3.3, the stability threshold value is

$$\frac{mrK}{eH} = 0.66.$$

Figure 4 shows a typical pair of simulation results where: (a) $c_1 = 0.4$, $c_2 = 0.25$, and $A = \max m(t) = 0.65 < 0.66$; (b) $c_1 = 0.8$, $c_2 = 0.13$, and $a_1 = \min m(t) = 0.67 > 0.66$. Again the numerical results are consistent with the analytical predictions.

4. Conclusions and discussion

We have presented new results to the Floquet theory which facilitate the calculation of the Floquet exponents in several nontrivial cases. In particular, we have established that the Floquet exponents can be directly computed from the diagonal entries when the coefficient matrices are triangular, and the Floquet exponents are eigenvalues of a constant matrix when the coefficient matrix commutes with its antiderivative matrix. We apply these results to the study of the stability of periodic solutions to linear and nonlinear periodic systems, as well as delay linear periodic systems. We also apply our findings in Floquet theory to the analysis of a new cholera model to explore threshold values of cholera epidemics with seasonal oscillations.

The study of periodic solutions to linear and nonlinear periodic systems is both important and challenging, and there are many open questions. For example, given a linear periodic system $\dot{X}(t) = A(t)X(t)$, $X \in R^n$, under what conditions can we transform it to a triangular periodic system? For another example, consider the Floquet exponents of the delay linear system (10) with general periodic matrix functions $B_i(t)$'s. Does the formula in Equation (13) still hold? Consider all delay linear periodic triangular systems with the same dimension, $\dot{X}(t) = L(t, X_t)$. These L_t's form a nest algebra. One interesting question one wants to ask is how this nest algebra is related to the Banach space C. We know the evolution operator for each delay linear periodic triangular system is still "triangular". Are these operators form a nest algebra over the Banach space? If yes, is it unique, and is there a basis? Answers to these questions may reveal deeper structures of solutions to delay linear periodic systems.

Acknowledgements

The authors are grateful to Michael Li at University of Alberta for helpful discussion. The authors would also like to thank the anonymous referees for helpful comments.

Funding

J.P. Tian and J. Wang acknowledge partial support from the National Science Foundation under [grant number 1216907] and [grant number 1216936], respectively.

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