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# Evolution Algebras Generated by Gibbs Measures

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**Abstract**—In this article we study algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras. We give a constructive description of associating evolution algebras to the function spaces (cell spaces) defined by graphs and state spaces and Gibbs measure $\mu$. For finite graphs we find some evolution subalgebras and other useful properties of the algebras. We obtain a structure theorem for evolution algebras when graphs are finite and connected. We prove that for a fixed finite graph, the function spaces has a unique algebraic structure since all evolution algebras are isomorphic to each other for whichever Gibbs measures assigned. When graphs are infinite graphs then our construction allows a natural introduction of thermodynamics in studying of several systems of biology, physics and mathematics by theory of evolution algebras.

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## 1. INTRODUCTION

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics etc.

In book [6] the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed, with pointers to some further research topics.

Gibbs measures are familiar subjects in various areas of applied probability. Originally they were defined in the framework of lattice spin systems to describe thermodynamic equilibrium states [3–5].

In the present paper we explore some algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras. The study of these function spaces defined by graphs and state spaces equipped with Gibbs measures was inspired by Mendelian genetics. For finite graphs we find some evolution subalgebras and other useful properties of the algebras. We obtain a structure theorem for evolution algebras when graphs are finite and connected. We also prove that for a fixed finite graph, the function spaces (cell spaces) has a unique algebraic structure since all evolution algebras are isomorphic each other for whichever Gibbs measures assigned. Note that each evolution algebra can be uniquely determined by an assignment of its structural coefficients which can be arranged into a quadratic matrix. For infinite graphs we must use limit Gibbs measures, then one of the key problems is to determine the entries (structural coefficients) of the matrices which are already infinite sizes; the second problem is to investigate the evolution algebras which correspond to these matrices. These investigations allows to a natural introduction of thermodynamics

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in the study of the evolution algebras. More precisely, if graph $G$ is infinite (countable) then one can associate a limit Gibbs measure $\mu$ by a Hamiltonian $H$ (defined on $G$) and temperature $T > 0$ [3–5]. It is known that depending on the Hamiltonian and the values of the temperature the measure $\mu$ can be non unique. In this case there is a phase transition of the physical system with the Hamiltonian $H$. We ask: how the thermodynamics (the phase transition) will affect properties of evolution algebras corresponding to Gibbs measures of the Hamiltonian $H$, and how evolution algebras reflect thermodynamics. We will give some comments about this problem.

This paper is organized as follows. In Section 2, we give some preliminaries about graphs, evolution algebras, and Gibbs measures. In Section 3, For finite graphs and finite state spaces, evolution algebras are constructed. When graphs are finite and connected, the structure theorem is obtained. We also study how Gibbs measures affect evolution algebras. In Section 4, we present two simple examples to illustrate the theorems in Section 3. In Section 5, we constructively define evolution algebras when graphs are countable. We also give an example to show that the limit evolution algebra for a given graph and a series of Gibbs measures is not unique. We list several open problems at the end.

## 2. PRELIMINARIES

**Graphs:** A graph $G$ is an ordered pair of disjoint sets $(\Lambda, L)$ such that $L$ is a subset of the set $\Lambda^{(2)}$ of unordered pairs of $\Lambda$. The set $\Lambda$ is the set of vertices and $L$ is the set of edges. An edge $\{x, y\}$ is said to join the vertices $x$ and $y$. If $\{x, y\} \in L$, then $x$ and $y$ are called neighboring vertices of $G$. We say that $G' = (\Lambda', L')$ is a subgraph of $G = (\Lambda, L)$ if $\Lambda' \subset \Lambda$ and $L' \subset L$. A graph is connected if for every pair $x, y$ of distinct vertices there is a path from $x$ to $y$. A maximal connected subgraph is a component of the graph.

**Evolution algebras:** Let $(A, \cdot)$ be an algebra over a field $K$. If it admits a basis $e_1, e_2, \ldots$, such that

$$e_i \cdot e_j = 0, \quad \text{if} \quad i \neq j,$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \quad \text{for any} \quad i,$$

we then call this algebra an evolution algebra.

Evolution algebras have following elementary properties (see [6])

—Evolution algebras are not associative, in general.

—Evolution algebras are commutative, flexible.

—Evolution algebras are not power-associative, in general.

—Direct sums of evolution algebras are also evolution algebras.

—Kronecker products of evolutions algebras are also evolution algebras.

**Function spaces (cell spaces) and Hamiltonian:** Let $G = (\Lambda, L)$ be a graph, and $\Phi$ be a state space with finite many states. For any $A \subseteq \Lambda$, a cell $\sigma_A$ on $A$ is defined as a function $x \in A \to \sigma_A(x) \in \Phi = \{1, 2, \ldots, q\}$; the set of all cells coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_\Lambda$ and $\sigma = \sigma_\Lambda$.

The energy of the cell $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = \sum_{A \subset \Lambda} I(\sigma_A) \tag{1}$$

where $I(\sigma_A) : \Omega_A \to R$ is a given potential function.

For a finite domain $D \subset \Lambda$ with the boundary condition $\varphi_{D^c}$ given on its complement $D^c = \Lambda \setminus D$, the conditional Hamiltonian is

$$H(\sigma_D \big| \varphi_{D^c}) = \sum_{A \subset \Lambda : A \cap D \neq \emptyset} I(\sigma_A), \tag{2}$$

where

$$\sigma_A(x) = \begin{cases} \sigma(x) & \text{if} \quad x \in A \cap D \\ \varphi(x) & \text{if} \quad x \in A \cap D^c. \end{cases}$$

**Gibbs measures:** We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega$ generated by cylinder subsets; all probability measures are considered on $(\Omega, \mathcal{B})$. A probability measure $\mu$ is called a Gibbs measure (with Hamiltonian $H$) if it satisfies the DLR equation: for all finite $D \subset \Lambda$ and $\sigma_D \in \Omega_D$:

$$\mu\left(\left\{\omega \in \Omega : \omega\big|_D = \sigma_D\right\}\right) = \int_\Omega \mu(\mathrm{d}\varphi)\nu_\varphi^D(\sigma_D), \tag{3}$$

where $\nu_\varphi^D$ is the conditional probability:

$$\nu_\varphi^D(\sigma_D) = \frac{1}{Z_{D,\varphi}} \exp\left(-\beta H\left(\sigma_D\big|\varphi_{D^c}\right)\right). \tag{4}$$

Here $\beta = \frac{1}{T}$, $T > 0$ temperature and $Z_{D,\varphi}$ stands for the partition function in $D$, with the boundary condition $\varphi$:

$$Z_{D,\varphi} = \sum_{\widetilde{\sigma}_D \in \Omega_D} \exp\left(-\beta H\left(\widetilde{\sigma}_D\big|\varphi_{D^c}\right)\right).$$

## 3. CONSTRUCTION OF EVOLUTION ALGEBRAS FOR FINITE GRAPHS

Let $G = (\Lambda, L)$ be a finite graph without loops and multiple edges. Furthermore, let $\Phi$ be a finite set. Denote by $\Omega$ the set of all cells $\sigma : \Lambda \to \Phi$.

One interpretation of the set $\Omega^2 = \Omega \times \Omega$ could be the set of all pairs of "parents".

Let $\{\Lambda_i, i = 1, \dots, m\}$ be the set of all maximal connected subgraphs (components) of the graph $G$.

For any $M \subset \Lambda$ and $\sigma \in \Omega$ denote $\sigma(M) = \{\sigma(x) : x \in M\}$. We say $\sigma(M)$ is a subcell iff $M$ is a maximal connected subgraph of $G$.

Fix two cells $\varphi, \psi \in \Omega$, (i.e. fix a pair of parents $\theta = (\varphi, \psi)$) and set

$$\Omega_\theta = \{\omega \in \Omega : \omega(\Lambda_i) = \varphi(\Lambda_i) \quad \text{or} \quad \omega(\Lambda_i) = \psi(\Lambda_i) \quad \text{for all} \quad i = 1, \dots, m\}. \tag{5}$$

**Remark 1.** The set $\Omega_\theta$ can be interpreted as the set of all possible children of the pair of parents $\theta$. A child $\omega$ can be born from $\theta$ if it only consists the subcells of its parents $\theta$. Such a set was first considered in [1] and in the general form in [2].

Denote $\Omega_\theta^2 = \{\psi = (\psi_1, \psi_2) : \psi_1, \psi_2 \in \Omega_\theta\}$. This set can be interpreted as a set of all possible pairs of parents which can be generated from the parents $\theta \in \Omega^2$.

Let $S(\Omega)$ be the set of all probability measures defined on the finite set $\Omega$. Now let $\mu \in S(\Omega)$ be a probability measure defined on $\Omega$ such that $\mu(\sigma) > 0$ for any cell $\sigma \in \Omega$; i.e $\mu$ is a Gibbs measure with some potential [4].

Consider measure $\mu^2 = \mu \times \mu$ on $\Omega^2$.

The (heredity) coefficients $P_{\varphi\psi}$ on $\Omega^2$ are defined as

$$P_{\varphi\psi} = \begin{cases} \frac{\mu^2(\psi)}{\mu^2({\Omega^2}_\varphi)}, & \text{if} \quad \psi \in {\Omega^2}_\varphi, \\ 0 \quad \text{otherwise.} \end{cases} \tag{6}$$

Obviously, $P_{\varphi\psi} \geq 0$ and $\sum_{\psi \in \Omega^2} P_{\varphi\psi} = 1$ for all $\varphi \in \Omega^2$.

Now we can define an evolution algebra $\mathcal{E} = \mathcal{E}(G, \Phi, \mu)$ generated by generators from $\Omega^2$ and following defining relations:

$$\begin{cases} \varphi^2 = \sum_{\psi \in \Omega^2} P_{\varphi\psi}\psi, \quad \text{for} \quad \varphi \in \Omega^2, \\ \varphi \cdot \psi = 0 \quad \text{if} \quad \varphi \neq \psi, \quad \varphi, \psi \in \Omega^2. \end{cases} \tag{7}$$

To study properties of the evolution algebra $\mathcal{E}$ we shall use properties of the set ${\Omega^2}_\sigma$.

**Lemma 3.1.** *If $\tau \in {\Omega^2}_\sigma$ then ${\Omega^2}_\tau \subseteq {\Omega^2}_\sigma$, where $\sigma \in \Omega^2$.*

**Proof.** Let $G$ be a (finite) graph and $\{\Lambda_i, i = 1, \dots, m\}$ the set of all maximal connected subgraphs of $G$. Denote by $\Phi_i = \Phi^{\Lambda_i}$ the set of all subcells defined on $\Lambda_i$, $i = 1, \dots, m$. Then any cell $\rho \in \Omega$ has representation $\rho = (\rho_1, \dots, \rho_m)$, with $\rho_i \in \Omega_i$, $i = 1, \dots, m$.

For $\sigma = (\varphi, \psi) \in \Omega^2$ with $\varphi = (\varphi_1, \dots, \varphi_m)$, $\psi = (\psi_1, \dots, \psi_m) \in \Omega$, we have

$$\Omega^2_\sigma = \{\tau = (\nu, \omega) = ((\nu_1, \dots, \nu_m), (\omega_1, \dots, \omega_m)) : \nu_i, \omega_i \in \{\varphi_i, \psi_i\}, i = 1, \dots m\}.$$

Take $\tau = (\nu, \omega) \in \Omega^2_\sigma$ then $\nu_i, \omega_i \in \{\varphi_i, \psi_i\}$ for any $i = 1, \dots, m$. Thus for arbitrary $\xi = (\xi^1, \xi^2) \in \Omega^2_\tau$ we have $\xi^1_i, \xi^2_i \in \{\nu_i, \omega_i\} \subseteq \{\varphi_i, \psi_i\}$ for any $i = 1, \dots, m$, i.e. $\xi \in \Omega^2_\sigma$. This completes the proof. □

As a corollary of Lemma 3.1 we have

**Lemma 3.2.** *If $\tau \in \Omega^2_\sigma$ and $\sigma \in \Omega^2_\tau$, then $\Omega^2_\sigma = \Omega^2_\tau$.*

Note that for any $\sigma \in \Omega^2$ we have $\sigma \in \Omega^2_\sigma$. The next lemma gives condition on $\sigma$ under which $\Omega^2_\sigma$ contains only $\sigma$ i.e., $\Omega^2_\sigma = \{\sigma\}$.

**Lemma 3.3.** *$\Omega^2_\sigma = \{\sigma\}$ if and only if $\sigma = (\varphi, \psi)$ with $\varphi = \psi$.*

**Proof.** By definition of $\Omega^2_\sigma$ with $\sigma = (\varphi, \psi)$ we have $(\varphi, \varphi), (\varphi, \psi), (\psi, \varphi), (\psi, \psi) \in \Omega^2_\sigma$. This completes the proof. □

**Theorem 3.1.** *The set $\Omega^2_\sigma$ generates an evolution subalgebra of the evolution algebra $\mathcal{E}$ for any $\sigma \in \Omega^2$.*

**Proof.** Take $\varphi \in \Omega^2_\sigma$. By (6) and (7) we get

$$\varphi^2 = \sum_{\psi \in \Omega^2} P_{\varphi\varphi\psi} \psi = \sum_{\psi \in \Omega^2_\sigma} P_{\varphi\psi} \psi.$$

According to Lemma 3.1 we have $\Omega^2_\varphi \subseteq \Omega^2_\sigma$. Then if we denote the subalgebra that is generated by $\Omega^2_\sigma$ by $\langle \Omega^2_\sigma \rangle$, it is clear that $\varphi^2 \in \langle \Omega^2{}_\sigma \rangle$, whenever $\varphi \in \Omega^2_\sigma$. Thus $\Omega^2_\sigma$ generates an evolution algebra. Theorem is proved. □

As a corollary of Theorem 3.1 and Lemma 3.3 we have a corollary.

**Corollary 3.1.** *The evolution algebra $\mathcal{E}$ has the evolution subalgebra $\langle \theta \rangle$ which is generated by one generator $\theta = (\varphi, \varphi)$ for any $\varphi \in \Omega$.*

**Remark 2.** 1) Note that $\langle \theta \rangle$ is an one dimensional subalgebra and number of such algebras is equal to $|\Omega| = |\Phi|^{|\Lambda|}$ where $|M|$ denotes number of elements (cardinality) of $M$.

2) For any $\sigma = (\varphi, \psi) \in \Omega^2$ with $\varphi \neq \psi$ to construct minimal set $\Omega^2_\sigma$ we must take a connected graph $G$. Then $\Omega^2_\sigma$ contains four elements only.

**Remark 3.** hFor any $\sigma = (\varphi, \psi) \in \Omega^2$ with $\varphi \neq \psi$ we have $\omega_1 = (\varphi, \varphi), \omega_2 = (\psi, \psi) \in \Omega^2_\sigma$. By Lemma 3.3 we get $\Omega^2_{\omega_i} = \{\omega_i\}$, $i = 1, 2$.

We put these remarks together, and have a structural theorem for finite connected graphs.

**Theorem 3.2.** *If the connected graph $G = (\Lambda, L)$ is finite and $|\Lambda| = n$, then $\mathcal{E} = \mathcal{E}(G, \Phi, \mu)$ has dimension of $n^{2k}$, where $k = |\Phi|$. Furthermore, $\mathcal{E}(G, \Phi, \mu)$ has $\frac{1}{2} n^k (n^k - 1)$ 4-dimensional evolution subalgebras, and $n^k$ 1-dimensional evolution subalgebras. $\mathcal{E}(G, \Phi, \mu)$ has a two-level hierarchy, 0-th level has $n^k$ 1-dimensional subalgebras and 1st level has $\frac{1}{2} n^k (n^k - 1)$ 2-dimensional subalgebras.*

For a fixed graph $G$, when we take different Gibbs measures for the function space $\Omega$, the algebra $\mathcal{E} = \mathcal{E}(G, \Phi, \mu)$ will be different. What are relations among these algebras? or How do Gibbs measures affect algebras? The following theorem answers this question when the graph $G$ is finite.

**Theorem 3.3.** *For a fixed finite graph $G = (\Lambda, L)$, all evolution algebras $\mathcal{E}(G, \Phi, \mu)$ defined by different Gibbs measures $\mu$ are isomorphic each other. When G is connected, their hierarchies have two levels, share the same skeleton.*

**Proof.** Take two Gibbs measures $\mu_1$ and $\mu_2$, let's assign a 1-1 and onto map $\digamma$ between $\mathcal{E}(G, \Phi, \mu_1)$ and $\mathcal{E}(G, \Phi, \mu_2)$. These two algebras share the same generator set $\Omega = \{\varphi | \varphi : \Lambda \longrightarrow \Phi\}$. We define $\digamma(\varphi) = \varphi$, and then linearly extend this map to the whole algebra. Let's denote, the structural coefficients for the algebra defined by the Gibbs measure $\mu_i$ by $P_{\varphi\psi}(\mu_i)$. From the properties of Gibbs measures, $P_{\varphi\psi}(\mu_1) \neq 0$ if and only if $P_{\varphi\psi}(\mu_2) \neq 0$, or $P_{\varphi\psi}(\mu_1) = 0$ if and only if $P_{\varphi\psi}(\mu_2) = 0$. Then the map $\digamma$ keeps all generating relations, and keep all algebraic relations too. $\digamma$ is an algebraic map. Therefore, $\mathcal{E}(G, \Phi, \mu_1)$ is isomorphic to $\mathcal{E}(G, \Phi, \mu_2)$ [6].

When $G$ is connected, from Theorem 3.2 we know that the hierarchy of the algebra $\mathcal{E}(G,\Phi,\mu)$ has two levels for any Gibbs measure $\mu$. As to these algebra's common skeleton which is homomorphic to each of them [6], it keeps all algebraic relations except the dimensions. So, in the 0-th level the skeleton has $n^k$ 1-dimensional subalgebras and in the 1st level the skeleton has $\frac{1}{2}n^k(n^k-1)$ 1-dimensional subalgebras. □

Actually, for these algebras $\mathcal{E}(G,\Phi,\mu)$ differed by Gibbs measure $\mu$, when $G$ is finite and connected graph, they represent a similar dynamical system. For example, there are two flows directs from each subalgebra at 1st level of their hierarchy to two different subalgebras at 0-th level of the hierarchy. We will illustrate this point in the following Example 1.

For two elements interpreted as parents $\sigma,\tau\in\Omega^2$ we denote $\tau\prec\sigma$ if $P_{\sigma\tau}>0$ (see formula (6)). By our construction $\tau\prec\sigma$ if and only if $\tau\in\Omega^2_\sigma$.

**Theorem 3.4.** *For any $\sigma\in\Omega^2$ there exists $n\in\{1,2,\dots,|\Omega^2_\sigma|\}$ and $\tau_1,\tau_2,\dots,\tau_n\in\Omega^2_\sigma$ such that $\tau_n\prec\tau_{n-1}\prec\dots\prec\tau_1\prec\sigma$ with $|\Omega^2_{\tau_i}|>2$, $i=1,\dots,n-1$ and $\Omega^2_{\tau_n}=\{\tau_n\}$.*

**Proof.** If $\Omega^2_\sigma=\{\sigma\}$ then $n=1$ and $\tau_1=\sigma$. If there exists $\tau_1\in\Omega^2_\sigma$ such that $\tau_1\neq\sigma$ then consider $\Omega^2_{\tau_1}$ which is subset (by Lemma 3.1) of $\Omega^2_\sigma$. If there exists $\tau_2\in\Omega^2_{\tau_1}$ such that $\tau_2\neq\tau_1,\sigma$ then $\Omega^2_{\tau_2}\subseteq\Omega^2_{\tau_1}$ and so on. Iterating this argument at most $|\Omega^2_\sigma|$ time, we get (see Remark 3) $\Omega^2_{\tau_n}=\{\tau_n\}$ with some $n\in\{1,\dots,|\Omega^2_\sigma|\}$. □

**Corollary 3.2.** *For any $\sigma\in\Omega^2$ there are subalgebras $\langle\Omega^2_{\tau_i}\rangle$, $i=1,\dots,n$ such that*

$$\langle\Omega^2_{\tau_n}\rangle\subseteq\langle\Omega^2_{\tau_{n-1}}\rangle\subseteq\dots\subseteq\langle\Omega^2_\sigma\rangle.$$

Recall the hierarchy structure theorem of evolution algebras that, for any element of the algebra, there is a sequence of subalgebras which ends at a simple subalgebra such that each of them contains this element. For the algebras $\mathcal{E}(G,\Phi,\mu)$, this corollary tells how the flows direct or where their children will be.

## 4. EXAMPLES

**Example 1:** Consider $\Lambda=\{1,2\}$, $L=\{\langle 1,2\rangle\}$, i.e. $G$ is connected graph with one edge $\langle 1,2\rangle$. Take $\Phi=\{a,A\}$ as a set of two alleles for some genetic trait, then

$$\Omega=\{\sigma_1=(a,a),\sigma_2=(a,A),\sigma_3=(A,a),\sigma_4=(A,A)\}.$$

$$\Omega^2=\{\varphi_{ij}=(\sigma_i,\sigma_j):i,j=1,2,3,4\},\quad \Omega^2_{\varphi_{ii}}=\{\varphi_{ii}\},$$

$$\Omega^2_{\varphi_{ij}}=\{\varphi_{ii},\varphi_{ij},\varphi_{ji},\varphi_{jj}\},\quad i,j=1,2,3,4.$$

Consider a Gibbs measure $\mu$ on $\Omega$: $\mu(\sigma_i)=p_i>0$, with $p_1+p_2+p_3+p_4=1$. Then we have

$$P_{\varphi_{ii}\psi}=\begin{cases}1 & \text{if}\quad \psi=\varphi_{ii},\\ 0 & \text{if}\quad \psi\neq\varphi_{ii}\end{cases}\qquad i=1,2,3,4,$$

and

$$P_{\varphi_{ij}\psi}=\begin{cases}p_i^2(p_i+p_j)^{-2} & \text{if}\quad \psi=\varphi_{ii},\\ p_j^2(p_i+p_j)^{-2} & \text{if}\quad \psi=\varphi_{jj},\\ p_ip_j(p_i+p_j)^{-2} & \text{if}\quad \psi=\varphi_{ij},\psi_{ji}.\end{cases}\qquad i\neq j,\quad i,j=1,2,3,4.$$

Correspondingly, the evolution algebra $\mathcal{E}_1=\mathcal{E}(G,\Phi,\mu)$ is given by relations

$$\begin{cases}\varphi_{ii}^2=\varphi_{ii},\quad i=1,2,3,4\\ \varphi_{ij}^2=(p_i+p_j)^{-2}\left(p_i^2\varphi_{ii}+p_ip_j(\varphi_{ij}+\varphi_{ji})+p_j^2\varphi_{jj}\right),\quad i\neq j,\quad i,j=1,\dots,4.\\ \varphi\cdot\psi=0\quad\text{if}\quad\varphi\neq\psi.\end{cases}\tag{8}$$

Denote the subalgebra generated by an element $\varphi$ by $\langle\varphi\rangle$. The algebra $\mathcal{E}_1$ has 6 subalgebras, $\langle\varphi_{12}\rangle$, $\langle\varphi_{13}\rangle$, $\langle\varphi_{14}\rangle$, $\langle\varphi_{23}\rangle$, $\langle\varphi_{24}\rangle$, and $\langle\varphi_{34}\rangle$. These algebras are not simple, and each has dimension of 4. For example, $\langle\varphi_{12}\rangle = \langle\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}\rangle$ has 4 generators, and its dimension also is 4. There are four persistent generators, $\langle\varphi_{ii}\rangle$, $i = 1, 2, 3, 4$. Each of these persistent elements generates a simple evolution algebra with dimension 1. According to [6], algebras $\langle\varphi_{11}\rangle$, $\langle\varphi_{22}\rangle$, $\langle\varphi_{33}\rangle$ and $\langle\varphi_{44}\rangle$ are 0-th simple subalgebras. The 1st subalgebras are generated by transit elements. There are 6 transit generators. For example, $\langle\varphi_{12}\rangle_1 = \langle\varphi_{12}, \varphi_{21}\rangle_1$. Therefore, this algebra $\mathcal{E}$ has two levels in its hierarchy, four simple subalgebras at the 0-th, six simple subalgebras at the 1-st level. Namely,

$$\mathcal{E}_1 = \langle\varphi_{11}\rangle \bigoplus \langle\varphi_{22}\rangle \bigoplus \langle\varphi_{33}\rangle \bigoplus \langle\varphi_{44}\rangle \dot{+} B_0,$$

$$B_0 = \langle\varphi_{12}\rangle \bigoplus \langle\varphi_{13}\rangle \bigoplus \langle\varphi_{23}\rangle \bigoplus \langle\varphi_{14}\rangle \bigoplus \langle\varphi_{24}\rangle \bigoplus \langle\varphi_{34}\rangle.$$

There are two dynamical flows from each subalgebra at the 1st level to two different subalgebras at the 0-th level. For example, from the subalgebra $\langle\varphi_{12}\rangle$, one flow directs to the subalgebra $\langle\varphi_{11}\rangle$ and the other one directs to the subalgebra $\langle\varphi_{22}\rangle$.

**Example 2:** Consider graph $G = (\Lambda, L)$ with $\Lambda = \{1, 2\}$, $L = \emptyset$ and $\Phi = \{a, A\}$. Then $\Omega^2$, $\Omega^2_{\varphi_{ii}}$, $\Omega^2_{\varphi_{ij}}$ are the same as the corresponding sets in Example 1 if $(ij) \neq (14), (41), (23), (32)$. But

$$\Omega^2_{\varphi_{14}} = \Omega^2_{\varphi_{41}} = \Omega^2_{\varphi_{23}} = \Omega^2_{\varphi_{32}} = \Omega^2,$$

which are large than corresponding sets in the Example 1.

Correspondingly, the evolution algebra $\mathcal{E}_2 = \mathcal{E}(G, \Phi, \mu)$ is given by relations

$$\begin{cases} \varphi^2_{ii} = \varphi_{ii}, \quad i = 1, 2, 3, 4 \\ \varphi^2_{ij} = (p_i + p_j)^{-2} \left( p_i^2 \varphi_{ii} + p_i p_j (\varphi_{ij} + \varphi_{ji}) + p_j^2 \varphi_{jj} \right), \quad i \neq j, \quad (ij) \neq (14), (41), (23), (32) \\ \varphi^2_{23} = \varphi^2_{32} = \varphi^2_{14} = \varphi^2_{41} = \sum_{i,j=1}^4 p_i p_j \varphi_{ij}, \quad (ij) = (14), (41), (23), (32) \\ \varphi \cdot \psi = 0 \quad \text{if} \quad \varphi \neq \psi. \end{cases}$$

The $\mathcal{E}_2$ has 3 levels in its hierarchy showed in the following.

$$\mathcal{E}_2 = \langle\varphi_{11}\rangle \bigoplus \langle\varphi_{22}\rangle \bigoplus \langle\varphi_{33}\rangle \bigoplus < \varphi_{44} > \dot{+} B_0,$$

$$B_0 = \langle\varphi_{12}\rangle \bigoplus \langle\varphi_{13}\rangle \bigoplus \langle\varphi_{24}\rangle \bigoplus \langle\varphi_{34}\rangle \dot{+} B_1,$$

$$B_1 = \langle\varphi_{14}\rangle.$$

Dynamically, there are 4 flows from the subalgebra $\langle\varphi_{14}\rangle$ at the second level direct to each subalgebra at the 1st level. There are two flows from each subalgebra at the 1st level direct two different subalgebras at the 0-th level.

**Remark 4.** If we assume $e_1 = \varphi_{44}$, $e_2 = \varphi_{11}$, $e_3 = \varphi_{24} = \varphi_{42} = \varphi_{34} = \varphi_{43}$, $e_4 = \varphi_{12} = \varphi_{21} = = \varphi_{13} = \varphi_{31}$, $e_5 = \varphi_{14} = \varphi_{41}$, $e_6 = \varphi_{22} = \varphi_{33} = \varphi_{23} = \varphi_{32}$ and $p_2 = p_3$ then the evolution algebra is given by relations

$$\begin{cases} e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_6^2 = e_6 \\ e_3^2 = (p_2 + p_4)^{-2} \left( p_4^2 e_1 + 2p_2 p_4 e_3 + p_2^2 e_6 \right), \\ e_4^2 = (p_1 + p_2)^{-2} \left( p_1^2 e_2 + 2p_1 p_2 e_4 + p_2^2 e_6 \right), \\ e_5^2 = p_4^2 e_1 + p_1^2 e_2 + 4p_2 p_4 e_3 + 4p_1 p_2 e_4 + 2p_1 p_4 e_5 + 4p_2^2 e_6. \end{cases}$$

This algebra is similar to the algebra considered in Example 7 of ([6], p. 89), but does not coincide with it for any $p_i > 0, i = 1, ..., 4$ i.e for any Gibbs measure $\mu$.

## 5. THE CASE OF INFINITE GRAPHS

We consider countable graphs to have countable many vertices and edges. Let $G = (\Lambda, L)$ be a countable graph, and $\Phi$ be a finite set. Then the set of all functions $\sigma : \Lambda \to \Phi$, denoted by $\Omega$, is uncountable. Let $S(\Omega)$ be the set of all probability measures defined on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the standard $\sigma-$algebra generated by the finite-dimensional cylindrical set. Let $\mu$ be a measure on $(\Omega, \mathcal{F})$ such that $\mu(B) > 0$ for any finite-dimensional cylindrical set $B \in \mathcal{F}$. Note that only Gibbs measure have this property [4].

Fix a finite subset $M \subset \Lambda$. We say that $\sigma \in \Omega$ and $\varphi \in \Omega$ are equivalent if $\sigma(M) = \varphi(M)$. Let $\xi = \{\Omega_i, i = 1, 2, \dots, q\}$, (where $q = |\Phi|^{|M|}$) be the partition of $\Omega$ generated by this equivalent relation, $\Omega_i$ contains all equivalent elements.

Let $G_M = (M, L_M)$ be finite subgraph of $G$ with $L_M = \{\langle x, y\rangle \in L : x, y \in M\}$. Let $M_i, i = 1, \dots, m$ be maximal connected subgraphs of $G_M$.

Consider $\Omega^2 = \Omega \times \Omega$ as the set of all pairs of parents. Fix $\sigma = (\varphi, \psi) \in \Omega^2$ and set

$$\Omega_{M,\sigma} = \{\omega \in \Omega : \omega(M_i) = \sigma(M_i), \quad \text{or} \quad \omega(M_i) = \psi(M_i), \quad i = 1, \dots, m\}.$$

Define $P^M_{\varphi\psi}$ on $\Omega^2_{M,\varphi}$ as

$$P^M(\varphi, \psi) = \begin{cases} \frac{\mu^2(\Omega_i^2)}{Z^M_{\varphi,i}}, & \text{if} \quad \psi \in \Omega^2_{M,\varphi} \cap \Omega_i^2, \quad i = 1, \dots, q, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where

$$Z^M_{\varphi,i} = \mu^2(\Omega^2_{M,\varphi} \cap \Omega_i^2) \sum_{j=1}^{q} \mu^2(\Omega_j^2).$$

Using (9) we now can define an evolution algebra $\mathcal{E}_M$ by the following defining relations

$$\varphi \cdot \varphi = \int_{\Omega^2} P(\varphi, \psi)\psi = \sum_{i=1}^{q} \frac{\mu^2(\Omega_i^2)}{Z^M_{\varphi,i}} \int_{\Omega^2_{M,\varphi}\cap\Omega_i^2} \psi, \quad \varphi \in \Omega^2$$

$$\varphi \cdot \psi = 0 \quad \text{if} \quad \varphi \neq \psi, \quad \varphi, \psi \in \Omega^2.$$

Note that by our construction the coefficients (9) depend on fixed $M$. Consider an increasing sequence of connected, finite sets $M_1 \subset M_2 \subset \dots \subset M_n \subset \dots$. such that $\cup_n M_n = \Lambda$. For each $M_n$, using Gibbs measure $\mu_n$ defined on $\Omega_n = \Phi^{M_n}$, we can define evolution algebras $\mathcal{E}_{M_n}$, $n = 1, 2, \dots$

Denote by $P^{(n)}_{\varphi\psi}$ the coefficients (9) which is constructed by $M_n$ and $\mu_n$. An interesting problem is to describe the set of all cells $\varphi, \psi$ such that the following limits exist

$$P_{\varphi\psi} = \lim_{n\to\infty} P^{(n)}_{\varphi\psi}. \tag{10}$$

Note that if $\varphi$ and $\psi$ are equal almost sure (i.e. the set $\{x : \varphi(x) \neq \psi(x)\}$ is finite) then the limit (10) exists.

The evolution algebra $\mathcal{E} = \mathcal{E}(\mu)$ defined by limit coefficients (10) and a limit Gibbs measure $\mu$ (see section 2) is called a limit evolution algebra.

These investigations allows to a natural introduction of thermodynamics in studying of such evolution algebras. More precisely, if $\Omega$ is continual set then one can associate the Gibbs measure $\mu$ by a Hamiltonian $H$ (defined on $\Omega$) and temperature $T > 0$. It is known that depending on the Hamiltonian and the values of the temperature the measure $\mu$ can be non unique. In this case there is a phase transition of the physical system with the Hamiltonian $H$. From the constructive definition of evolution algebras, each $\mathcal{E}_{M_n}$ is a finite dimensional algebra, but the limit evolution algebra will be an algebra with uncountable dimension if it exists. Phase transitions in thermodynamics could bring out some ideas in study continual algebras.

Let us consider the Potts model on $Z^d$ as an example of Hamiltonian $H$. In this case $\Omega$ is the set of all cells $\sigma : Z^d \to \Phi = \{1, \dots, q\}$.

The (formal) Hamiltonian of the Potts model is

$$H(\sigma) = -J \sum_{\substack{x,y \in Z^d \\ ||x-y||=1}} \delta_{\sigma(x)\sigma(y)},$$

where $J \in R$.

It is well-known (see [5], Theorem 2.3) that for the Potts model with $q \geq 2$ there exist critical temperature $T_{\rm cr}$ such that for any $T < T_{\rm cr}$ there are $q$ distinct extreme Gibbs measures $\mu^{(i)}$, $i = 1, \dots, q$.

Let $M_n, n = 1, 2, \dots$ are fixed (as above) subsets of $Z^d$. Note that, for $T$ low enough each measure $\mu^{(i)}$ is a small deviation of the constant cells $\sigma^{(i)} \equiv i, i = 1, \dots, q$. This means that, when $T \to 0$, the restriction $\mu_n^{(i)}$ of $\mu^{(i)}$ on $M_n$ converges (as $n \to \infty$) weakly towards $\delta_{\sigma^{(i)}}$, the Dirac measure at the constant cell $\sigma^{(i)}$. Thus $P_{\varphi\psi}^{(n)} \to 1$, as $n \to \infty$ if $\varphi, \psi$ are equal to $\sigma^{(i)}$ almost sure and $P_{\varphi\psi}^{(n)} \to 0$ otherwise. Hence, when $T \to 0$ then Gibbs measures $\mu^{(i)}$, $i = 1, \dots, q$ of the Potts model give $q$ different *limit* evolution algebras $\mathcal{E}^{(i)} = \langle \sigma^{(i)} \rangle$, $i = 1, \dots, q$.

**Open problems:** We have defined evolution algebras $\mathcal{E} = \mathcal{E}(G, \Phi, \mu)$ for a given graph, state spaces and Gibbs measures. If graphs are finite and connected, we obtained a structure theorem for this type of algebras. If graphs are finite but not connected, we don't know their precise structures. If graphs are countable, we do not know any deep results. We therefore list several interesting open problems here.

1. If graphs are finite and not connected, what are structures of $\mathcal{E}(G, \Phi, \mu)$? For example, if a graph has two components, how many simple algebras does $\mathcal{E}(G, \Phi, \mu)$ have? How does its hierarchy look like?

2. If graphs are countable and connected, what are the structures of algebras $\mathcal{E}(G, \Phi, \mu)$? Do we have a similar theorem as that in the case of finite graph Theorem 3.2?

3. For a fixed countable graph, how do Gibbs measures affect the limit evolution algebras defined in this section? From the example on Potts model above we know there could be several or many different limit evolution algebras. Are these limit algebras isomorphic or homomorphic? Do we have a similar theorem as the Theorem 3.3?

4. For countable graphs, is there any condition under which the limit evolution algebra exist?

5. More generally, for countable graphs how does the thermodynamics (the phase transition) affect properties of evolution algebras corresponding to Gibbs measures of the Hamiltonian $H$? How do evolution algebras reflect thermodynamics?

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