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GENERALIZED LONG-MOODY REPRESENTATIONS
OF BRAID GROUPS

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Dedicated to the memory of Professor Xiao-Song Lin

Long and Moody give a method of constructing representations of the braid groups B_n. We discuss some ways to generalize their construction. One of these gives representations of subgroups of B_n, including the Gassner representation of the pure braid group as a special case. Another gives representations of the Hecke algebra.

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1. Introduction

The braid group B_n is the group with presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \,|\, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1, \ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1 \rangle.$$

It has a rich and interesting representation theory [2] [3].

D. D. Long, in joint work with J. Moody, gave a construction that inputs a representation of a certain semidirect product $F_n \rtimes B_n$ and outputs a new representation of B_n. At first this construction may not sound useful, since its input seems to be harder to come by than its output. However there are at least two factors in its favor. First, the output seems to be in some sense more sophisticated than the input. For example, one can start with a one dimensional representation and obtain the Burau representation. Second, the semidirect product $F_n \rtimes B_n$ is

a subgroup of B_{n+1}. Thus one can recursively apply the algorithm to construct a sequence of increasingly complicated representations of B_n for all n.

Let F_n be the free group $\langle g_1, \dots, g_n \rangle$. There is a well known action of B_n on F_n, which gives rise to a semidirect product $F_n \rtimes B_n$. This requires an arbitrary choice of convention. In this paper, the relations are as follows.

- $g_{i+1}\sigma_i = \sigma_i g_i$,
- $g_i\sigma_i = \sigma_i g_i g_{i+1} g_i^{-1}$,
- $g_j\sigma_i = \sigma_i g_j$ if $j \notin \{i, i+1\}$.

The following is Theorem 2.1 in [9], which Long credits to be a joint work with Moody.

Theorem 1.1. *Given a representation $\rho\colon F_n \rtimes B_n \to \mathrm{GL}(V)$ we may construct a representation $\rho^+\colon B_n \to \mathrm{GL}(V^{\oplus n})$.*

The method of construction can be summarized as follows. The representation ρ determines a system of local coefficients on the n-times punctured disk D_n. The first cohomology of D_n with these local coefficients then turns out to be isomorphic to $V^{\oplus n}$. The braid group B_n is the mapping class group of D_n, so it has a natural induced action on this vector space. For details, see [9].

Long also describes how to explicitly compute the matrices for ρ^+ in terms of the matrices for ρ. In this paper we will only use this more concrete (but less well motivated) definition.

In Section 2, we give a proof of Theorem 1.1. In Section 3, we generalize the Long-Moody construction to subgroups of B_n. We then discuss important special cases of this construction. In Section 4, we apply the construction to the pure braid group P_n, obtaining the Gassner representation as a special case. In Section 5, we describe how these methods can be used to obtain Lawrence's representations. These are of interest because they contain all representations of the Hecke algebra. Finally, in Section 6, we give the "reduced" version of the Long-Moody construction.

For convenience, we will work with finite dimensional representations over the field $\mathbf{C}$. The construction works equally well for infinite dimensional representations, and for representations over any field or commutative ring.

Note that, while the main results of this paper are stated as Theorems and Corollaries and so on, they are really *constructions*. Taken literally, a statement of the form "Given one representation, we may construct another representation" is trivial. Of course, the interesting thing is the method of construction.

2. The Universal Long-Moody Representation

The aim of this section is to give a proof of Theorem 1.1. Our proof is not exactly new, but it does make explicit an idea that is implicit in [9]. This is the idea of a representation we call the "universal Long-Moody representation".

Proof of Theorem 1.1. Recall that we are given a representation $\rho\colon F_n \rtimes B_n \to$ $\mathrm{GL}(V)$, and must construct a representation $\rho^+\colon B_n \to \mathrm{GL}(V^{\oplus n})$. First we define the "universal Long-Moody representation" of B_n. This is a homomorphism from B_n to the group of invertible n by n matrices with entries in the group ring of $F_n \rtimes B_n$.

Let $F_n \rtimes B_n$ be the semidirect product as defined earlier. Let $\mathbf{Z}[F_n \rtimes B_n]$ denote the group ring of this group. For $i = 1, \ldots, n-1$, let R_i be the following two by two matrix with entries in $\mathbf{Z}[F_n \rtimes B_n]$.

$$R_i = \begin{bmatrix} 0 & g_i \\ 1 & 1 - g_i \end{bmatrix}.$$

This is invertible.

$$R_i^{-1} = \begin{bmatrix} (1 - g_i^{-1}) & 1 \\ g_i^{-1} & 0 \end{bmatrix}.$$

Let ϕ be the following homomorphism from B_n to the group of invertible n by n matrices with entries in $\mathbf{Z}[F_n \rtimes B_n]$.

$$\phi(\sigma_i) = \sigma_i \begin{bmatrix} I_{i-1} & & \\ & R_i & \\ & & I_{n-i-1} \end{bmatrix}.$$

Here, I_{i-1} and I_{n-i-1} are the appropriately sized identity matrices, and σ_i acts on the matrix by scalar multiplication on the left.

It remains to check that ϕ satisfies the braid relations. The most difficult is the braid relation

$$\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i) = \phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1})$$

One can verify this by direct calculation, using the relations in $F_n \rtimes B_n$.

Having defined ϕ, it is now easy to construct ρ^+. Choose a finite basis for V. (This is not strictly necessary - we leave it to the reader to fill in the details when V is infinite dimensional.) The image of ρ now consists of square matrices. For any $\beta \in B_n$, let $\rho^+(\beta)$ be the block matrix obtained by applying ρ to each entry of $\phi(\beta)$. This is an element of $\mathrm{GL}(V^{\oplus n})$, as required. $\qquad\square$

The simplest nontrivial example of this construction is stated in the following proposition.

Proposition 2.1. *If $\rho\colon F_n \rtimes B_n \to \mathbf{C}^*$ is a one dimensional representation then $\rho^+\colon B_n \to \mathrm{GL}(n, \mathbf{C})$ is the unreduced Burau representation, up to specializing and rescaling.*

Proof. Suppose $\rho\colon F_n \rtimes B_n \to \mathbf{C}^*$ is a one dimensional representation. The generators σ_i of B_n are all conjugate to each other, so must all be mapped to the same

value $s \in \mathbf{C}^*$. Similarly, the generators g_i are all conjugate to each other in $F_n \rtimes B_n$, so must all be mapped to the same value $t \in \mathbf{C}^*$. Then

$$\rho^+(\sigma_i) = s \begin{bmatrix} I_{i-1} & & & \\ & 0 & t & \\ & 1 & 1-t & \\ & & & I_{n-i-1} \end{bmatrix}.$$

This is the familiar matrix of the unreduced Burau representation, rescaled by a factor of s.   □

3. The Long-Moody Construction for Subgroups of B_n

Suppose G is a subgroup of B_n. Then $F_n \rtimes G$ is a subgroup of $F_n \rtimes B_n$. The aim of this section is to prove the following theorem.

Theorem 3.1. *Given a subgroup G of B_n and a representation $\rho\colon F_n \rtimes G \to \mathrm{GL}(V)$, we may construct a representation $\rho^+\colon G \to \mathrm{GL}(V^{\oplus n})$.*

The construction is the same as that of Long and Moody. Given an element $\beta \in G$, we let $\rho^+(\beta)$ be the result of applying ρ to each of the entries in the matrix $\phi(\beta)$. We need only check that these entries lie in the domain of ρ. Thus it remains only to prove the following lemma.

Lemma 3.2. *If $\beta \in B_n$ then every entry of the matrix $\phi(\beta)$ is of the form $a\beta$ for some $a \in \mathbf{Z}[F_n]$.*

Proof. Write β as a word in the generators σ_i. Recall that $\phi(\sigma_i)$ is the scalar σ_i times a matrix with entries in $\mathbf{Z}[F_n]$. Thus $\phi(\beta)$ is a product of terms, each of which is either a scalar $\sigma_i^{\pm 1}$ or a matrix with entries in $\mathbf{Z}[F_n]$. The product of the terms $\sigma_i^{\pm 1}$ in this product, taken in order, is equal to β.

Observe that if A is any matrix with entries in $\mathbf{Z}[F_n]$ and σ_i is a generator of B_n then $\sigma_i A$ is equal to $A'\sigma_i$ for some matrix A' with entries in $\mathbf{Z}[F_n]$. To see this, apply the relations of $F_n \rtimes B_n$ to each term in each entry of A. By using this observation repeatedly, we can move the terms $\sigma_i^{\pm 1}$ to the right of the product expression for $\phi(\beta)$. We obtain an expression of the form $A\beta$, where A is a matrix with entries in $\mathbf{Z}[F_n]$.   □

4. The Gassner Representation

Let P_n denote the pure braid group. The aim of this section is to prove the following theorem.

Theorem 4.1. *Given a representation $\rho\colon P_{n+1} \to \mathrm{GL}(V)$ we may construct an n parameter family of representations $\rho^+_{t_1,\dots,t_n}\colon P_n \to \mathrm{GL}(V^{\oplus n})$.*

Long achieved a similar result by different methods in [7] and [8]. The construction there uses an action of the braid group by diffeomorphisms on a certain

representation variety. To obtain a linear representation, he takes the derivative of this action at any point that is fixed by the action of B_n, or P_n. Long was most interested in the Gassner representation, but the approach is quite general, and could give a result similar to Theorem 4.1.

Our construction is more along the lines of [9], especially the following theorems, which are [9, Theorem 2.4] and [9, Corollary 2.6] respectively.

Theorem 4.2. *Given a representation ρ of B_{n+1}, we may construct a new representation ρ^+ of B_n.*

Theorem 4.3. *Given a representation ρ of $F_n \rtimes B_n$, we may construct a one parameter family of representations ρ_t^+ of B_n.*

The first follows immediately from the fact that $F_n \rtimes B_n$ is a subgroup of B_{n+1}. The second uses ρ to define a one parameter family of representations ρ_t of $F_n \rtimes B_n$, and then applies the Long-Moody construction to these. The representations ρ_t are just "rescalings" of ρ, but the resulting family of representations ρ_t^+ is often more interesting than ρ^+.

To prove Theorem 4.1, we apply the same ideas in the context of the pure braid group.

Proof of Theorem 4.1. Let F_n be the subgroup of P_{n+1} consisting of all braids in which the first n strands are straight. This is the free group with generators

$$g_i = (\sigma_n \ldots \sigma_{i+1})\sigma_i^2(\sigma_n \ldots \sigma_{i+1})^{-1}.$$

Let P_n be a subgroup of P_{n+1} in the usual way. It is well known that P_n and F_n together generate P_{n+1}. Indeed, P_{n+1} is isomorphic to the subgroup $F_n \rtimes P_n$, of our semidirect product $F_n \rtimes B_n$.

Fix $t_1, \ldots, t_n \in \mathbf{C}^*$. Let $\rho_{t_1,\ldots,t_n} : P_{n+1} \to \mathrm{GL}(V)$ be the representation such that

$$\rho_{t_1,\ldots,t_n}(g_i) = t_i\rho(g_i)$$

for any of the generators g_i of F_n, and

$$\rho_{t_1,\ldots,t_n}(\beta) = \rho(\beta)$$

for any $\beta \in P_n$.

To see that $\rho_{t_1,\ldots,t_n}$ is well defined, note that if $\beta \in P_n$ and $i \in \{1, \ldots, n\}$ then

$$\beta g_i \beta^{-1} = w g_i w^{-1}$$

for some $w \in F_n$. These relations determine the semidirect product structure of P_{n+1}, and they are preserved by $\rho_{t_1,\ldots,t_n}$.

Now simply apply Theorem 3.1 to $\rho_{t_1,\ldots,t_n}$ to obtain the desired representation of P_n. $\qquad\square$

As an example, we compare the results of Theorems 4.3 and 4.1 when ρ is the trivial representation. If ρ is the trivial representation of $F_n \rtimes B_n$ then Theorem 4.3 gives the unreduced Burau representation ρ_t^+ of B_n (by Proposition 2.1). If ρ is the trivial representation of P_{n+1} then $\rho_{t_1,\ldots,t_n}^+$ is the unreduced Gassner representation of P_n.

For more sophisticated examples, we could think of Theorem 4.1 as giving a "Gassner" version of representations of B_n other than the Burau representation.

5. Lawrence's Construction

The Hecke algebra is a certain quotient of the group algebra $\mathbf{C}B_n$. For our purposes, a representation of the Hecke algebra is any representation ρ of B_n such that the matrices $\rho(\sigma_i)$ all satisfy a quadratic relation. Such representations are the subject of a large body of ongoing research [10] [1].

The following proposition suggests a connection between the representation ϕ from Section 2 and representations of the Hecke algebra.

Proposition 5.1. $(\phi(\sigma_i) + \sigma_i g_i I)(\phi(\sigma_i) - \sigma_i I) = 0.$

This can be checked by direct calculation. However it does not qualify as a representation of the Hecke algebra, since the "scalars" of this quadratic relation lie in a non-commutative ring.

By rescaling, it suffices to consider representations of B_n that satisfy

$$(\rho(\sigma_i) + qI)(\rho(\sigma_i) - I) = 0$$

for some $q \in \mathbf{C}^*$. For generic values of q, such representations of B_n can be enumerated by the partitions of n. In [5], Lawrence gives a topological construction of the representations of the Hecke algebra that correspond to partitions of n of the form $(n - m, m)$. This was generalized to all partitions in [6].

In [9, Corollary 2.10], Long describes how his construction can be used to parallel that of [5]. The methods in this section are intended to parallel those of [6] (which had not appeared at the time of [9]). We will not prove any concrete connection. To do that would require a thorough treatment of the homological details behind the Long-Moody construction. Instead, this section could be viewed as a self contained description of a way to construct representations of the Hecke algebra.

Let $B_{n,m}$ denote the set of braids in B_{n+m} such that the first n nodes at the bottom are connected to the first n nodes at the top, and (hence) the last m nodes at the bottom are connected to the last m nodes at the top. This is generated by

$$\sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \ldots, \sigma_{n+m-1}.$$

Lemma 5.2. *Given a representation $\rho\colon B_{n,m+1} \to \mathrm{GL}(V)$, we may construct a representation $\rho^+\colon B_{n,m} \to \mathrm{GL}(V^{\oplus(n+m)})$.*

Proof. Let $B_{n,m}$ be a subgroup of $B_{n,m+1}$ in the obvious way. Let F_{n+m} be the set of braids in $B_{n,m+1}$ such that the first $n + m$ strands are straight. This is the

free group with generators

$$g_i = (\sigma_{n+m} \ldots \sigma_{i+1})\sigma_i^2(\sigma_{n+m} \ldots \sigma_{i+1})^{-1}.$$

Then $B_{n,m}$ and F_{n+m} together generate a subgroup $F_{n+m} \rtimes B_{n,m}$ of $B_{n,m+1}$. The result now follows from Theorem 3.1. $\qquad\square$

Theorem 5.3. *Given a representation $\rho\colon B_m \to \mathrm{GL}(V)$, we may construct a one parameter family of representations $\rho_q^+\colon B_n \to \mathrm{GL}(V^{\oplus k})$, where $q \in \mathbf{C}^*$ and $k = (n+m-1)(n+m-2)\ldots(n+1)(n)$.*

Proof. Suppose we are given $\rho\colon B_m \to \mathrm{GL}(V)$ and $q \in \mathbf{C}^*$. Let $\rho'\colon B_{n,m} \to \mathrm{GL}(V)$ be the representation such that

- $\rho'(\sigma_i) = 1$ for $i = 1, \ldots, n-1$,
- $\rho'(\sigma_n^2) = q$,
- $\rho'(\sigma_{n+i}) = \rho(\sigma_i)$ for $i = 1, \ldots, m-1$.

Now apply Lemma 5.2 to ρ' recursively m times to obtain the desired representation of B_n. $\qquad\square$

For example, suppose ρ is the one dimensional representation of B_2 given by $\rho(\sigma_1) = t$, where $t \in \mathbf{C}^*$. Then ρ_q^+ is the Lawrence-Krammer representation [5] [4].

By analogy to [6], we expect the representation ρ_q^+ to be especially interesting in when

$$(\rho(\sigma_i) - q^{-1}I)(\rho(\sigma_i) - I) = 0.$$

However Lawrence still has to do some more work to obtain a representation of the Hecke algebra as a subrepresentation of ρ_q^+. Rather than try to imitate this in our purely algebraic setting, we give an alternative construction that gives a representation of the Hecke algebra directly.

Theorem 5.4. *Given $m \le n$ and a representation $\rho\colon B_m \to \mathrm{GL}(V)$, we can construct a representation $\rho^+\colon B_n \to \mathrm{GL}(V \oplus \ldots \oplus V)$, where there are $\binom{n}{m}$ copies of V in the direct sum. If ρ satisfies the relation $(\rho(\sigma_i) + qI)(\rho(\sigma_i) - I) = 0$ then so does ρ^+.*

Proof. We start by constructing a "universal" Lawrence representation. Let $\langle q \rangle$ denote the infinite cyclic group generated by a formal variable q. Our universal Lawrence representation will be a homomorphism ϕ from B_n to the group of invertible $\binom{n}{m}$ by $\binom{n}{m}$ matrices with entries in $\mathbf{Z}[\langle q \rangle \times B_m]$. Enumerate the rows and columns of this matrix by basis vectors $e_{a_1,\ldots,a_m}$ for $1 \le a_1 < \ldots < a_m \le n$.

To make the definition more readable, we first fix some notation. Let $\mathbf{a}$ denote a sequence of the form $a_1, \ldots, a_j$, where $1 \le a_1 < \ldots < a_j < i$. Let $\mathbf{b}$ denote a

sequence of the form $b_1, \ldots, b_k$, where $i + 1 < b_1 < \ldots < b_k \le n$. In each of the following, assume that j and k are such that the length of the whole sequence is n.

$$\phi(\sigma_i)(e_{\mathbf{a},\mathbf{b}}) = e_{\mathbf{a},\mathbf{b}}$$
$$\phi(\sigma_i)(e_{\mathbf{a},i,\mathbf{b}}) = e_{\mathbf{a},i+1,\mathbf{b}}$$
$$\phi(\sigma_i)(e_{\mathbf{a},i+1,\mathbf{b}}) = qe_{\mathbf{a},i,\mathbf{b}} + (1-q)e_{\mathbf{a},i+1,\mathbf{b}}$$
$$\phi(\sigma_i)(e_{\mathbf{a},i,i+1,\mathbf{b}}) = \sigma_{j+1}e_{\mathbf{a},i,i+1,\mathbf{b}}.$$

We must check that ϕ satisfies the braid relations, so that it gives a well defined representation of B_n. This is done by direct computation and checking cases. We will only describe some special cases that give a flavor of the general proof.

If $m = 0$ then ϕ is the trivial representation.

If $m = 1$ then ϕ is the unreduced Burau representation.

If $m = n$ then ϕ maps each braid β to the one by one matrix $[\beta]$.

Finally, consider the case $m = 2$ and $n = 3$. The matrices for ϕ with respect to the basis $(e_{2,3}, e_{1,3}, e_{1,2})$ are as follows.

$$\phi(\sigma_1) = \begin{bmatrix} 1-q & 1 & 0 \\ q & 0 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix},$$

$$\phi(\sigma_2) = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 1-q & 1 \\ 0 & q & 0 \end{bmatrix}.$$

One can check that these matrices satisfy the braid relation $ABA = BAB$.

To prove that ϕ satisfies the braid relations in general, we can break down the image into blocks, each of which resembles one of the above cases.

Having defined ϕ, we can define the specific representation ρ^+ in the same way as we did for the Long-Moody construction. First choose a basis for V, and an ordering for our basis vectors $e_{a_1,\ldots,a_k}$. The images of ρ and ϕ can then be written as square matrices. Given $\beta \in B_n$, let $\rho^+(\beta)$ be the block matrix obtained by applying ρ to each entry of $\phi(\beta)$ and substituting a value $q \in \mathbf{C}^*$.

By some further computation and checking of cases, every nonzero entry of the matrix

$$(\phi(\sigma_i) + qI)(\phi(\sigma_i) - I)$$

is of the form

$$(q + \sigma_j)(1 - \sigma_j)$$

for some j. Now suppose ρ satisfies the relation

$$(\rho(\sigma_i) + qI)(\rho(\sigma_i) - I) = 0.$$

If we apply ρ to each entry of $(\phi(\sigma_i)+qI)(\phi(\sigma_i)-I)$ then we obtain the zero matrix. Thus ρ^+ satisfies the same quadratic relation as ρ. $\quad\square$

6. The Reduced Long-Moody Construction

The unreduced Burau representation obtained in Proposition 2.1 is reducible. It is the direct sum of a one dimensional representation with the reduced Burau representation. In [9, Theorem 2.11], Long shows that this happens in general. We record here how to compute the "reduced" version of the Long-Moody construction.

For $i = 1, \ldots, n-1$, let S_i be the following three by three matrix with entries in $\mathbf{Z}[F_n \rtimes B_n]$.

$$S_i = \begin{bmatrix} 1 & g_i & 0 \\ 0 & -g_i & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is invertible.

$$S_i^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -g_i^{-1} & 0 \\ 0 & g_i^{-1} & 1 \end{bmatrix}.$$

Let ϕ_r be the following homomorphism from B_n to the group of invertible $n-1$ by $n-1$ matrices with entries in $\mathbf{Z}[F_n \rtimes B_n]$.

$$\phi_r(\sigma_i) = \sigma_i \begin{bmatrix} I_{i-2} & & \\ & S & \\ & & I_{n-i-2} \end{bmatrix}.$$

If i is 1 or $n-1$ then part of S is "cut off" by the edge of the matrix. In other words,

$$\phi_r(\sigma_1) = \sigma_1 \begin{bmatrix} -g_1 & 0 & \\ 1 & 1 & \\ & & I_{n-3} \end{bmatrix},$$

$$\phi_r(\sigma_{n-1}) = \sigma_{n-1} \begin{bmatrix} I_{n-3} & & \\ & 1 & g_{n-1} \\ & 0 & -g_{n-1} \end{bmatrix}.$$

Acknowledgment

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