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On Several types of universal invariants of framed links and 3-manifolds derived from Hopf algebras

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Abstract

Without using representations of quasitriangular ribbon Hopf algebras, Hennings, Kauffman and Radford, Ohtsuki and the author gave different methods to construct invariants of links and 3-manifolds respectively. To understand these different methods and the resultant invariants, we made a comprehensive comparison study in this paper. We show the relations among the universal invariants of framed links defined by these different authors. We also figured out the relations among the resultant invariants of 3-manifolds defined by these different authors. Ignoring the difference by scalar constants and the inversion of Hopf algebras, we found that the invariants of 3-manifolds obtained by these authors are the same or equivalent to each other. Therefore, in a sense, there is only one way to construct invariants of 3-manifolds without using representation theory of Hopf algebras.

1. *Introduction*

Firstly Reshetikhin and Turaev [**7**], by using representations of quasitriangular ribbon Hopf algebras, found invariants of regular isotopy for coloured framed links. For a particular finite-dimensional quasitriangular ribbon Hopf algebra (the reduced quantum groups $U_q(sl_2)'$, where q is a root of unit), they found that a linear combination of these invariants is a constant under the Kirby moves and thus got invariants of the associated 3-manifolds obtained from the framed links by surgery. Hennings [**2**], however, did not use representations of quasitriangular ribbon Hopf algebras, and constructed general invariants of regular isotopy for coloured framed links. For unimodular finite-dimensional quasitriangular ribbon Hopf algebras, from their right integrals he derived a linear map which can be taken such that the resultant invariants unchangeable under the Kirby moves. In fact these linear maps enabled him to obtained a type of invariant of the associated 3-manifolds. Afterwards, Kauffman and Radford [**3**] also put forward a constructing procedure for unoriented links to obtain a type of invariant of the links and the associated 3-manifolds. Ohtsuki [**4**], without using representations of Hopf algebras, built universal invariants of framed links on the level of tensor; and then he defined a type of invariant of the corresponding 3-manifolds from those universal invariants of framed links. Following Kauffman and Radford [**3**], the author [9] has also given universal invariants of framed links and 3-manifolds. In this paper, we shall give the explicit relations among all these invariants of framed links and 3-manifolds for general unimodular finite dimensional quasitriangular ribbon Hopf algebras without representative theory involved.

The paper is organized as follows: in Section 2, we review some related properties of Hopf algebras and construct several propositions of special Hopf algebras. In Section 3, we derive a type of universal invariant of framed links from Hennings [2] and give the relation between this type of invariant and universal invariants of framed links introduced by Ohtsuki. In Section 4, we make the comparisons between universal invariants of framed links derived by Kauffman–Radford, Ohtsuki and from Hennings respectively. In Section 5, we discuss the algebraic representation of Kirby moves and give the relations among these several types of universal invariants of framed links and 3-manifolds invariants.

2. *Preliminaries*

Given any Hopf algebra H, if the antipode of H is invertible, we can have two other Hopf algebras, which accompany H. Notations about general Hopf algebras in this paper are as in Sweedler's book [8].

PROPOSITION 2·1. *If $(H, m, \Delta, \mu, \varepsilon, s)$ and $(H, m, \Delta', \mu, \varepsilon, s')$ are both Hopf algebras, then $s \circ s' = s' \circ s = id$ and so $s' = s^{-1}$, where H is an associative algebra over a field k with multiplication m and unit μ, Δ is a co-multiplication, ε is a co-unit and s is an antipode, $\Delta' = \tau \circ \Delta$.*

Proof. We know that $Hom(H^C, H^A)$ is a convolution algebra with multiplication $*$ and unit $\mu\varepsilon$. Taking any element $h \in H$, we have

$$(ss' * s)(h) = m(ss' \otimes s)\Delta(h) = \sum_{(h)} ss'(h_{(1)})s(h_{(2)}) = s\bigg(\sum_{(h)}(h_{(2)}s'(h_{(1)})\bigg).$$

We also have

$$(id * s')(h) = m(id \otimes s')\Delta'(h) = \sum_{(h)}(h_{(2)}s'(h_{(1)})) = \mu\varepsilon(h) = \varepsilon(h)1.$$

Therefore, we get

$$(ss' * s)(h) = s(\mu\varepsilon(h)) = \varepsilon(h)s(1) = \varepsilon(h)1 = \mu\varepsilon(h).$$

Similarly, we have

$$(s * ss')(h) = \mu\varepsilon(h).$$

Hence, ss' is the inverse of s under $*$. We also know that the inverse of s is id under $*$, so

$$ss' = id.$$

On the other hand,

$$\begin{aligned}(s's * s')(h) &= m(s's \otimes s')\Delta'(h) = \sum_{(h)} s's(h_{(2)})s'(h_{(1)}) \\ &= s'\bigg(\sum_{(h)} h_{(1)}s(h_{(2)})) = s'(\mu\varepsilon(h)\bigg) \\ &= \varepsilon(h)1 = \mu\varepsilon(h).\end{aligned}$$

Similarly,

$$(s' * s's)(h) = \mu\varepsilon(h).$$

It shows that $s's$ is the inverse of s' under $*$. But we know the inverse of s' is id under $*$. Therefore, $s's = id$.

Remark 1. In this proof, we, in fact, take advantage of two convolution algebras, one corresponding to the co-multiplication Δ; the other corresponding to the co-multiplication Δ'. From this proof, we can realize the following analogous proposition.

PROPOSITION 2·2. *If $(H, m, \Delta, \mu, \varepsilon, s)$ and $(H, m', \Delta, \mu, \varepsilon, s')$ are both Hopf algebras, then $ss' = s's = id$, where $m' = m\tau$, and τ is the permutation.*

In other words, if Hopf algebra $(H, m, \Delta, \mu, \varepsilon, s)$ has an invertible antipode, we immediately have two other Hopf algebras $(H, m, \Delta', \mu, \varepsilon, s^{-1})$ and $(H, m', \Delta, \mu, \varepsilon, s^{-1})$.

PROPOSITION 2·3. *Let $(H, m, \Delta, \mu, \varepsilon, s, R)$ be a quasitriangular Hopf algebra with invertible antipode and an invertible element $R \in H \otimes H$, called Yong–Baxter element. Then, $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1})$ and $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1})$ are both quasitriangular Hopf algebras.*

Proof. Let $R = \sum_i a_i \otimes b_i$, then

$$R^{-1} = (s \otimes id)R = \sum_i s(a_i) \otimes b_i$$

and

$$(\Delta \otimes id)R = R_{13}R_{23}, \qquad (id \otimes \Delta)R = R_{13}R_{12},$$
$$\Delta'(h) = R \cdot \Delta(h) \cdot R^{-1}, \qquad \forall h \in H.$$

It is easy to see that the following formulae are correct

$$R_{13}^{-1} = \sum_i s(a_i) \otimes 1 \otimes b_i,$$
$$R_{23}^{-1} = \sum_i 1 \otimes s(a_i) \otimes b_i,$$
$$R_{12}^{-1} = \sum_i s(a_i) \otimes b_i \otimes 1.$$

Now, let's verify that $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1})$ is a quasitriangular Hopf algebra. Firstly, we verify $(id \otimes \Delta')R^{-1} = R_{13}^{-1}R_{23}^{-1}$. From $(\Delta \otimes id)R = R_{13}R_{23}$, we have

$$\sum_{i,(a_i)} a_{i(1)} \otimes a_{i(2)} \otimes b_i = \sum_{j,k} a_j \otimes a_k \otimes b_j b_k \tag{2·1}$$

and

$$\begin{aligned}(\Delta' \otimes id)R^{-1} &= (\Delta' \otimes id)\Big(\sum_i s(a_i) \otimes b_i\Big)\\ &= \sum_i \Delta' s(a_i) \otimes b_i = \sum_i (\tau\Delta)s(a_i) \otimes b_i\\ &= \sum_i \tau(\Delta s)(a_i) \otimes b_i\end{aligned}$$

$$= \sum_i \tau(\tau(s \otimes s)\Delta(a_i)) \otimes b_i = \sum_i (s \otimes s)\Delta(a_i) \otimes b_i$$
$$= \sum_{i,(a_i)} s(a_{i(1)}) \otimes s(a_{i(2)}) \otimes b_i.$$

That is,

$$(\Delta' \otimes id)R^{-1} = \sum_{i,(a_i)} s(a_{i(1)}) \otimes s(a_{i(2)}) \otimes b_i. \tag{2·2}$$

After multiplication of R_{13}^{-1} and R_{23}^{-1}, we have

$$R_{13}^{-1}R_{23}^{-1} = \sum_{j,k} s(a_j) \otimes s(a_k) \otimes b_j b_k. \tag{2·3}$$

When we apply $s \otimes s \otimes id$ on both sides of equation (2·1), we obtain

$$\sum_{i,(a_i)} s(a_{i(1)}) \otimes s(a_{i(2)}) \otimes b_i = \sum_{j,k} s(a_j) \otimes s(a_k) \otimes b_j b_k. \tag{2·4}$$

Therefore, comparing equations (2·2), (2·3) and (2·4), we get

$$(\Delta' \otimes id)R^{-1} = R_{13}^{-1}R_{23}^{-1}.$$

Secondly, we verify $(id \otimes \Delta')R^{-1} = R_{13}^{-1}R_{12}^{-1}$. From $(id \otimes \Delta)R = R_{13}R_{12}$, we have

$$\sum_{i,(b_i)} a_i \otimes b_{i(1)} \otimes b_{i(2)} = \sum_{j,k} a_j a_k \otimes b_k \otimes b_j \tag{2·5}$$

and

$$(id \otimes \Delta')R^{-1} = (id \otimes \Delta')\left(\sum_i s(a_i) \otimes b_i\right)$$
$$= \sum_{i,(a_i)} s(a_i) \otimes b_{i(2)} \otimes b_{i(1)}. \tag{2·6}$$

After multiplication of R_{13}^{-1} and R_{12}^{-1},

$$R_{13}^{-1}R_{12}^{-1} = \sum_k s(a_k) \otimes 1 \otimes b_k \cdot \sum_j s(a_j) \otimes b_j \otimes 1$$
$$= \sum_{j,k} s(a_k)s(a_j) \otimes b_j \otimes b_k$$
$$= \sum_{j,k} s(a_j a_k) \otimes b_j \otimes b_k. \tag{2·7}$$

When we apply $(id \otimes \tau) \circ (s \otimes id \otimes id)$ on both sides of equation (2·5), we obtain

$$\sum_{i,(b_i)} s(a_i) \otimes b_{i(1)} \otimes b_{i(2)} = \sum_{j,k} s(a_j a_k) \otimes b_k \otimes b_j$$

and, then

$$\sum_{i,(b_i)} s(a_i) \otimes b_{i(2)} \otimes b_{i(1)} = \sum_{j,k} s(a_j a_k) \otimes b_j \otimes b_k. \tag{2·8}$$

Therefore, comparing equations (2·6), (2·7) and (2·8), we get

$$(id \otimes \Delta')R^{-1} = R_{13}^{-1} R_{12}^{-1}.$$

Now, from

$$\forall h \in H, \qquad \Delta'(h) = R \cdot \Delta(h) \cdot R^{-1},$$

we have

$$\Delta(h) = R^{-1} \cdot \Delta'(h) \cdot R.$$

Thus $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1})$ is a quasitriangular Hopf algebra. By analogous verifications, we know $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1})$ is also a quasitriangular Hopf algebra.

PROPOSITION 2·4. *Suppose $(H, m, \Delta, \mu, \varepsilon, s, R)$ is a quasitriangular Hopf algebra with an invertible antipode, and $Z(H)$ is its center. If there is an element $v \in Z(H)$, which satisfies*

$$v^2 = us(u), \qquad s(v) = v,$$
$$\varepsilon(v) = 1, \qquad \Delta(v) = \left(R_{21}^{-1} R_{12}^{-1}\right)^{-1} (v \otimes v),$$

where

$$u = u_H = \sum_i s(b_i) a_i, \qquad R = \sum_i a_i \otimes b_i.$$

In other words, if $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ is a ribbon Hopf algebra, then $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ and $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ are both ribbon Hopf algebras.

Proof. We firstly should determine the analogous element u' for $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1})$ as u_H for $(H, m, \Delta, \mu, \varepsilon, s, R)$. Since

$$R^{-1} = \sum_i s(a_i) \otimes b_i,$$

we could formally take

$$u' = \sum_i s^{-1}(b_i) s(a_i).$$

By Drinfeld [**1**]

$$u^{-1} = \sum_i b_i s^2(a_i), \tag{2·9}$$

we have

$$(u')^{-1} = \sum_i b_i s^{-2}(s(a_i)) = \sum_i b_i s^{-1}(a_i).$$

Since

$$(s \otimes s)R = R = (s^{-1} \otimes s^{-1})R,$$

we have

$$u' = \sum_i s^{-1}(b_i) s(a_i) = \sum_i s^{-1}(s(b_i)) s(s(a_i))$$
$$= \sum_i b_i s^2(a_i) = u^{-1}.$$

Secondly, we can determine v' as the following:

$$(v')^2 = u's^{-1}(u') = u^{-1}s^{-1}(u^{-1}).$$

Considering the equalities about u and u^{-1} (2·9), we have

$$s^{-1}(u^{-1}) = \sum_i s(a_i)s^{-1}(b_i)$$

and

$$\begin{aligned} s(u^{-1}) &= \sum_i s^3(a_i)s(b_i) \\ &= \sum_i s^3(s^{-2}(a_i))s(s^{-2}(b_i)) \\ &= \sum_i s(a_i)s^{-1}(b_i). \end{aligned}$$

Thus, we get

$$s^{-1}(u^{-1}) = s(u^{-1}).$$

Moreover, from $v^2 = us(u) = s(u)u$, we have

$$v^{-2} = u^{-1}s(u)^{-1} = u^{-1}s(u^{-1}).$$

Hence

$$v^{-2} = u^{-1}s^{-1}(u^{-1}) = (v')^2.$$

We therefore could take $v' = v^{-1}$. Then, it is natural that

$$\varepsilon(v)^{-1} = 1^{-1},$$

so

$$\varepsilon(v^{-1}) = 1,$$

and, by

$$s(v)^{-1} = v^{-1},$$

we have

$$v^{-1} = s^{-1}(v)^{-1}.$$

Finally, from

$$\Delta(v) = (R_{21}R_{12})^{-1}(v \otimes v),$$

we take the inverse, and get

$$\Delta(v)^{-1} = \Delta(v^{-1}) = (v^{-1} \otimes v^{-1})(R_{21}R_{12})$$

and

$$\begin{aligned} \Delta'(v)^{-1} &= \tau\Delta(v^{-1}) = \tau((v^{-1} \otimes v^{-1})(R_{21}R_{12})) \\ &= (v^{-1} \otimes v^{-1})(R_{12}R_{21}) = (v^{-1} \otimes v^{-1})\big(R_{21}^{-1}R_{12}^{-1}\big)^{-1} \\ &= \big(R_{21}^{-1}R_{12}^{-1}\big)^{-1}(v^{-1} \otimes v^{-1}). \end{aligned}$$

Hence, v^{-1} can be viewed as a ribbon element, and so then $(H, m, \Delta', \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ is a ribbon Hopf algebra. By an analogous verification, we can see $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ is also a ribbon Hopf algebra.

Remark 2. If there is a ribbon element in a quasitriangular Hopf algebra, ribbon element may not be unique.

PROPOSITION 2·5 (modified version in [**6**]). *Let* $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ *be a unimodular finite-dimensional ribbon Hopf algebra, then there exists a non-zero right integral* λ *such that*

$$(tr \otimes id)(\Delta(v^{-1}) \cdot G^{-1} \otimes 1) = \lambda(v^{-1})1, \tag{2·10}$$

$$(tr \otimes id)(\Delta(v) \cdot G^{-1} \otimes 1) = \lambda(v)1, \tag{2·11}$$

$$(tr \otimes id^{\otimes n})(\Delta^n(v^{-1}) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v^{-1})1^{\otimes n}, \tag{2·12}$$

$$(tr \otimes id^{\otimes n})(\Delta^n(v) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v)1^{\otimes n}, \tag{2·13}$$

where $tr = \lambda \cdot G$, $G = uv^{-1}$

Proof. We know that the following formulae are correct from [**6**]:

$$(u^{-1} \longleftarrow tr)u = \lambda(v^{-1})v$$

and

$$(s(u) \longleftarrow tr)s(u^{-1}) = \lambda(v)v^{-1}.$$

Then, we derive that,

$$\begin{aligned}(u^{-1} \longleftarrow tr)u &= ((tr \otimes id)(\Delta(u^{-1})))u \\ &= ((tr \otimes id)(\Delta(v^{-1}G^{-1})))Gv \\ &= ((tr \otimes id)(\Delta(v^{-1})\Delta(G^{-1})))Gv \\ &= (tr \otimes id)(\Delta(v^{-1}) \cdot (G^{-1} \otimes G^{-1}))Gv \\ &= ((tr \otimes id)(\Delta(v^{-1})G^{-1} \otimes 1))v \\ &= \lambda(v^{-1})v.\end{aligned}$$

Hence, from the last equality in the above, we get

$$(tr \otimes id)(\Delta(v^{-1}) \cdot G^{-1} \otimes 1) = \lambda(v^{-1})1_H.$$

Alternatively, this equation can be expressed as the following way:

$$(tr \otimes id)\Bigg(\sum_{(v^{-1})} v^{-1}_{(1)}G^{-1} \otimes v^{-1}_{(2)}\Bigg) = \lambda(v^{-1})1,$$

or,

$$\sum_{(v^{-1})} tr\big(v^{-1}_{(1)}G^{-1}\big)v^{-1}_{(2)} = \lambda(v^{-1})1. \tag{2·14}$$

Take the action of Δ on both sides of the equation (2·14),

$$\sum_{(v^{-1})} tr\big(v^{-1}_{(1)}G^{-1}\big) \sum_{(v^{-1}_{(2)})} v^{-1}_{(2)(1)} \otimes v^{-1}_{(2)(2)} = \lambda(v^{-1})1 \otimes 1.$$

Or, after taking into account the coassociativity, we have

$$(tr \otimes id \otimes id) \sum_{(v^{-1})} v^{-1}_{(1)} \otimes v^{-1}_{(2)} \otimes v^{-1}_{(3)} \cdot G^{-1} = \lambda(v^{-1})1 \otimes 1,$$

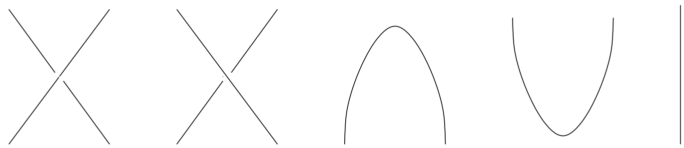

Fig. 1. Elementary diagrams.

again, write it simply,

$$(tr \otimes id \otimes id)(\Delta^{(2)}(v^{-1})G^{-1} \otimes 1 \otimes 1) = \lambda(v^{-1})1 \otimes 1.$$

Inductively, for any positive integer number n we have

$$(tr \otimes id^{\otimes n})(\Delta^{(n)}(v^{-1}) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v^{-1})1^{\otimes n}.$$

We also deduce that

$$\begin{aligned}(s(u) \longleftarrow tr)s(u^{-1}) & \\ &= ((tr \otimes id)(\Delta(s(u)))) \cdot s(v^{-1}G^{-1}) \\ &= ((tr \otimes id)(\Delta(s(Gv))))(Gv^{-1}) \\ &= ((tr \otimes id)(\Delta(vG^{-1})))(Gv^{-1}) \\ &= ((tr \otimes id)(\Delta(v) \cdot G^{-1} \otimes 1))v^{-1} \\ &= \lambda(v)v^{-1}.\end{aligned}$$

Hence, from the last equality in the above, we get

$$(tr \otimes id)(\Delta(v) \cdot G^{-1} \otimes 1) = \lambda(v)1.$$

Similarly, by induction, we have

$$(tr \otimes id^{\otimes n})(\Delta^{n}(v) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v)1^{\otimes n}.$$

The proof is completed.

Remark 3. In the constructions of universal invariants of links introduced by Hennings, Kauffman and Radford, respectively, there are certain formulae, which are similar to the above formulae. However, we prefer to use the above formulae since they are quite geometrically suggestive. Actually, we shall explain the relations between these formulae and Kirby moves in Section 5.

3. *A comparison between universal invariants of framed links introduced by Ohtsuki and that derived from Hennings*

In this section, we review two types of definitions of universal invariants of framed links. The difference between them will be pointed out.

According to [**4**] and [**2**], we give a diagrammatic formulation of these invariants and a computable method by using braid groups.

Any link diagram can be transformed topologically so that it is made up of combinations of the elementary diagrams in Figure 1.

Let L be a framed link and D be its diagram, which possesses m components $C_1, C_2, \ldots, C_m$. Given a finite-dimensional ribbon Hopf algebra $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ with $R = \sum_{i\in\Lambda} a_i \otimes b_i$ and $R^{-1} = \sum_{i\in\Lambda} a'_i \otimes b'_i$.

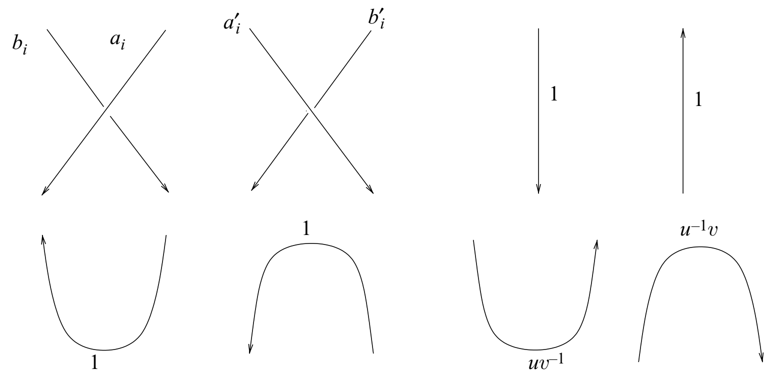

Fig. 2. The label rules of Hennings.

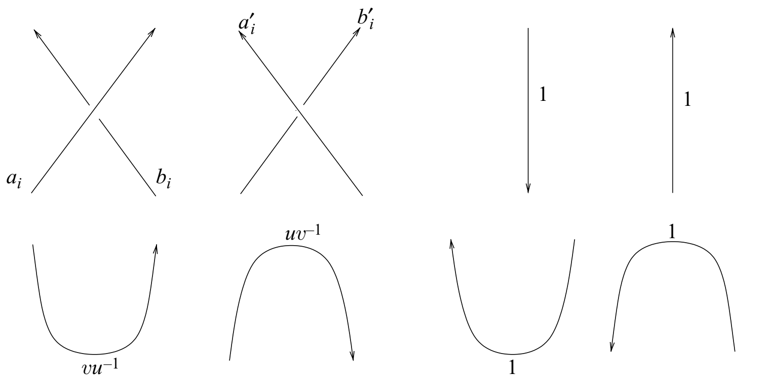

Fig. 3. The label rules of Ohtsuki.

Although Hennings has not specifically given so-called "universal invariants" of framed links, we can still obtain this type of invariants without using his elements $C_0 \subset H^*$ in his first step [**2**]. We label elementary diagrams according to the rules introduced by Hennings in Figure 2.

We can get a state for D, which is a map $\sigma : \{crosses\ of\ D\} \to \Lambda$. For each component, for example, jth, of D, choose a base point p_j and starting from p_j multiply all the labelled elements of jth component together in the order that are found according to the orientation of C_j. Then, we can obtain jth component of a weight $W_{He}(\sigma) \in H^{\otimes m}$. Because it is relevant to the choice of the base points, we must work modulo commutativity. After that, we can get an analogous definition,

$$\Phi_{He}(L) = \pi^{\otimes m}\left(\sum_{\sigma} W_{He}(\sigma)\right) \in (H/I)^{\otimes m}$$

where I is the linear subspace of H spanned by $xy - yx$ for any pair $x, y \in H$, and $\pi : H \to H/I$ is the nature projection.

THEOREM 3·1. *$\Phi_{He}(L)$ is an isotopic invariant of a framed link L derived from Hennings* [**2**].

The proof of this theorem is an analogy of [**4**, theorem 1·1]. We will therefore not give it here.

For comparison, we repeat the definition of Ohtsuki's in Figure 3.

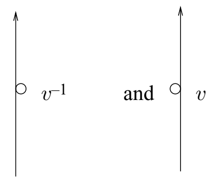

Fig. 4. Two oriented kinks.

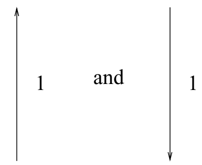

Fig. 5. Two oriented strings.

Because two oriented kinks in Figure 4 are derived from elementary properties of other labels, we may substitute them with two oriented strings as showed in Figure 5 as Ohtsuki did.

Denote Ohtsuki's weight and isotopy invariants by $W_O(\sigma)$ and $\Phi_O(L)$ respectively, so

$$\Phi_O(L) = \pi^{\otimes m}\left(\sum_{\sigma} W_O(\sigma)\right) \in (H/I)^{\otimes m}.$$

Now, we state the relation between isotopy invariants derived from Hennings' construction and that defined by Ohtsuki.

THEOREM 3·2.

$$\Phi_O(L) = S^{\otimes m}\Phi_{He}(L)$$

where $S\colon H/I \to H/I$ *is a linear map induced by antipode*: $S\colon H \to H$.

Proof. According to the properties of ribbon Hopf algebras and the above label definitions, it suffices to verify the theorem for the elementary diagrams. However, we would like to give another proof by using modified Alexander theorem [**4**].

Suppose that b is a framed braid, $b \in FB_n$ for some n and the closure $\widehat{b}$ of b be isotopic to L with m components.

Let S_n be the nth symmetric group. There is a natural map $\iota : B_n \to S_n$ by permuting the strings of the braid from the down end to the up end of the braid.

$\iota(b)$ can be divided into mutually disjointed cycles, and each cycle corresponding to a component of L. Given jth component C_j of L corresponds to the cycle $(k_1k_2\cdots k_l)$, suppose that $\sum a_{1i}\otimes a_{2i}\otimes\cdots\otimes a_{ni}$ is the "weight" of braid b according to Ohtsuki's definition, where $a_{ij} = d_{ij_1}d_{ij_2}\cdot\cdots\cdot d_{ij_{t_{ij}}}$, then the jth component of weight $W_O(\sigma)$ is

$$z_{(O)j} = \sum uv^{-1}a_{k_j i}\cdots uv^{-1}a_{k_1 i}.$$

Figure 6 shows how to obtain each component of Ohtsuki's weight. And now by taking the inverse direction of braid b, obtaining $\overline{b}$; by the definition of Ohtsuki's, we have

$$s(d_{ij_{t_{ij}}})\cdots s(d_{ij_2})s(d_{ij_1}) = s(d_{ij_1}d_{ij_2}\cdots d_{ij_{t_{ij}}}) = s(a_{ij}),$$

and also together, we have

$$\begin{aligned} s(z_{(O)j}) &= \sum s(a_{k_1 i})vu^{-1}s(a_{k_2 i})\cdots s(a_{k_l i})vu^{-1} \\ &= \sum s(uv^{-1}a_{k_l i}\cdots uv^{-1}a_{k_2 i}uv^{-1}a_{k_1 i}) =: \overline{z}_{(o)j}. \end{aligned} \tag{3·1}$$

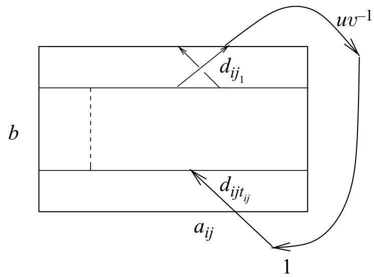

Fig. 6. Illustration for multiplication along cycles according to Ohtsuki's definition.

For $\overline{b}$, $\iota(\overline{b})$ must be the inverse of $\iota(b)$, so c_j corresponds to cycle $(k_l k_{l-1} \cdots k_2 k_1)$. Hence, by the definition of Hennings, we have

$$\overline{a}_{ij} = d_{ij_1} d_{ij_2} \cdots d_{ij_{t_{ij}}} = a_{ij}.$$

And, so together we have

$$\overline{z}_{(He)j} = \sum uv^{-1} a_{k_l i} \cdots uv^{-1} a_{k_2 i} uv^{-1} a_{k_1 i}. \tag{3·2}$$

Comparing (3·1) and (3·2), we have

$$s\big(\overline{z}_{(He)j}\big) = \overline{z}_{(O)j}.$$

In this way, we see for each component the theorem is right. Therefore,

$$\Phi_O(L) = S^{\otimes m} \Phi_{He}(L)$$

is correct for framed link L with m components.

PROPOSITION 3·1. *Let L be a framed link with m components and L' be L with the opposite orientation on the lth component, then*

$$\Phi_{He}(L') = \big(id^{\otimes(l-1)} \otimes S \otimes id^{\otimes(n-l)}\big)(\Phi_{He}(L)),$$

where we still denote by $S\colon H/I \to H/I$, the linear map induced by antipode: $S\colon H \to H$.

Proof. By Theorem 3·2 and a proposition of Φ_O in [**4**], we have the following equalities:

$$\Phi_O(L') = S^{\otimes m} \Phi_{He}(L'),$$

$$\Phi_O(L) = S^{\otimes m} \Phi_{He}(L)$$

and

$$\Phi_O(L') = \big(id^{\otimes(l-1)} \otimes S \otimes id^{\otimes(m-l)}\big) \Phi_O(L).$$

Put these three equalities together, we have

$$\begin{aligned} S^{\otimes m} \Phi_{He}(L') &= \big(id^{\otimes(l-1)} \otimes S \otimes id^{\otimes(m-l)}\big) S^{\otimes m} \Phi_{He}(L) \\ &= S^{\otimes m} \big(id^{\otimes(l-1)} \otimes S \otimes id^{\otimes(m-l)}\big) \Phi_{He}(L). \end{aligned}$$

Thus,

$$\Phi_{He}(L') = \big(id^{\otimes(l-1)} \otimes S \otimes id^{\otimes(m-l)}\big) \Phi_{He}(L).$$

The proof is complete.

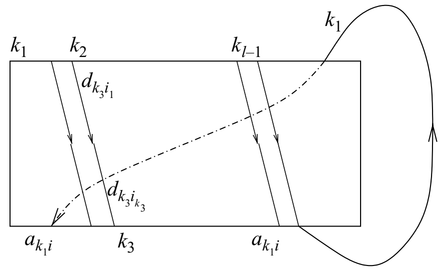

Fig. 7. Illustration for finding weights.

4. *A comparison between universal invariants of framed link derived from Hennings and that from Kauffman*

In [**3**], Kauffman and Radford do not directly give universal invariants of framed link. In [**9**] the author has given a definition $\theta(T)$. Because this type of universal invariant is constructed from unoriented link, they must be different from those invariants in Section 3. We can show the relations between them when we compute them for an oriented framed link.

Definition 4·1. $\rho_0 : (H, m, \Delta, \mu, \varepsilon, s, R, v) \to (H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ is defined by

$$\rho_0(\varepsilon) = \varepsilon, \quad \rho_0(s) = s^{-1}, \quad \rho_0(\Delta) = \Delta, \quad \rho_0(u) = u$$
$$\rho_0(m) = m', \quad \rho_0(R) = R^{-1}, \quad \rho_0(v) = v^{-1}.$$

In other words, ρ_0, We call this map the inversion, is a map that changes $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ into $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$.

THEOREM 4·1. *Let* $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ *be a unimodular finite dimensional ribbon Hopf algebra and L be an oriented framed link with n components, then*

$$\theta(L) = \rho(\Phi_{He}(L))$$

where θ *is as defined in* [**9**], ρ *is the map induced by* ρ_0.

Proof. As the proof of Theorem 3·2, let b be a framed braid, $b \in FB_p$ for some number p, and the closure $\widehat{b}$ of b be isotopic to L. Given jth component C_j of L corresponding to the cycle $(k_1 k_2 \cdots k_l)$, suppose that $\sum a_{1i} \otimes a_{2i} \otimes \cdots \otimes a_{ni}$ is the "weight" of braid b according to Hennings' definition, where $a_{ij} = d_{ij_1} d_{ij_2} \cdots d_{ij_{t_{ij}}}$, then

$$z_{(He)j} = \sum a_{k_1 i} u v^{-1} a_{k_2 i} \cdots a_{k_l i} u v^{-1}.$$

Now, forget the orientation of close braid $\widehat{b}$, label this unoriented $\widehat{b}$ according to [**9**] by using algebra $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$ and multiply algebraic elements upon each string, denoted by $\overline{a}_{ij} = \overline{d}_{ij_{t_{ij}}} \cdots \overline{d}_{ij_2} \overline{d}_{ij_1}$ move $\overline{a}_{k_l i}$ to up end of braid b, k_2, then the localized algebraic element of k_2 is $\overline{a}_{k_2 i} S^{-2}(\overline{a}_{k_l i})$. Move $\overline{a}_{k_2 i} S^{-2}(\overline{a}_{k_l i})$ to up end of braid b, k_3, then the localized algebraic element of k_3 is $\overline{a}_{k_3 i} S^{-2}(\overline{a}_{k_2 i}) S^{-4}(\overline{a}_{k_l i})$. Keep doing this as Figure 7 shows. Finally, when reaching where k_l is, we have the localized algebraic element $\overline{a}_{k_l i} S^{-2}(\overline{a}_{k_{l-1} i}) \cdots S^{-2(l-1)}(\overline{a}_{k_1 i})$. Thus,

$$\begin{aligned} z_{(\theta)j} &= \overline{a}_{k_l i} S^{-2}(\overline{a}_{k_{l-1} i}) \cdots S^{-2(l-1)}(\overline{a}_{k_1 i})(v u^{-1})^l \\ &= \overline{a}_{k_l i} v u^{-1} \overline{a}_{k_{l-1} i} v u^{-1} \cdots u v^{-1} \overline{a}_{k_1 i} v u^{-1}. \end{aligned}$$

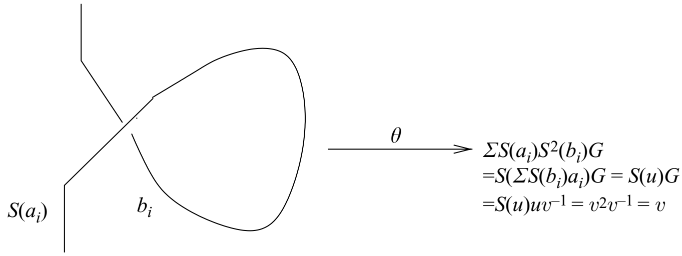

Fig. 8. Algebraic expression for a kink.

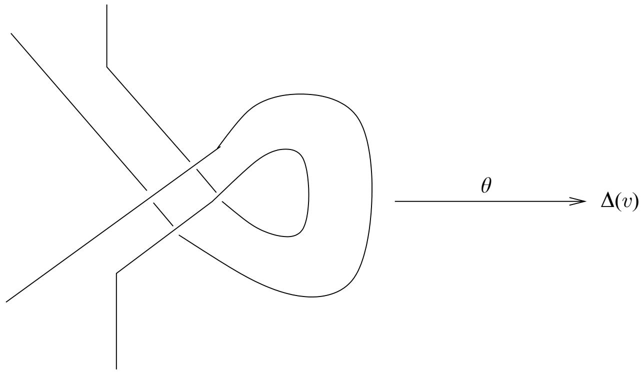

Fig. 9. Algebraic expression for a double kink.

In this way, for one component we have

$$z_{(\theta)j} = \rho\pi(z_{(He)j}).$$

Therefore, putting all components together,

$$\theta(L) = \rho(\Phi_{He}(L))$$

the proof is complete.

Remark 4. Since $\theta(L)$ is irrelevant to orientation of L, naturally, we can obtain

$$\theta(L) = \rho(\Phi_O(L)).$$

5. *The algebraic representation of Kirby Moves and relations among* 3*-manifold invariants derived from these universal link invariants*

When we construct an invariant of 3-manifold obtained by surgery along a family of links which transform mutually under Kirby moves, this invariant of 3-manifold must be constant to each of these links. In other words, it is invariant under Kirby moves. However, when we give the algebraic representation of Kirby moves, this question is changed completely into algebraic question.

By comparison from [**9**], we have an algebraic expression for a kink in Figure 8. By $\theta(\overline{\Delta}(T)) = \Delta(\theta(T))$ or by complicated computation, we have an algebraic expression for double kink as in Figure 9.

If we multiply $G^{-1} = vu^{-1}$ with some component of the tensor, we consider it as a result in closure of the component of the tangle correspondingly to the diagram. As examples of

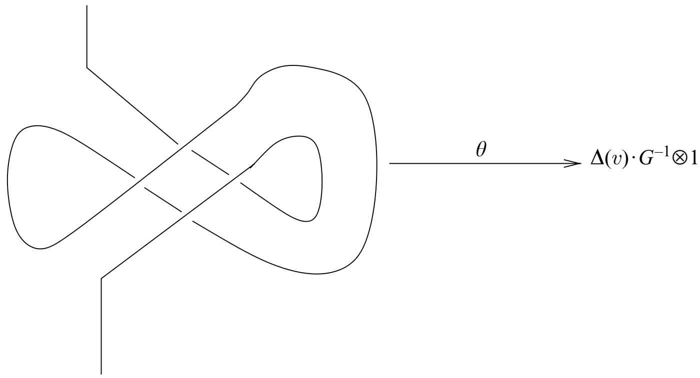

Fig. 10. Closure of Figure 9.

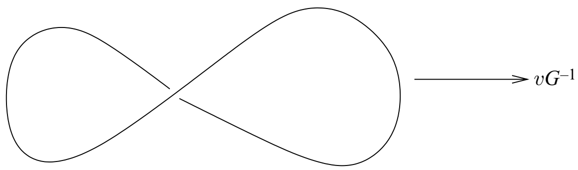

Fig. 11. Closure of Figure 8.

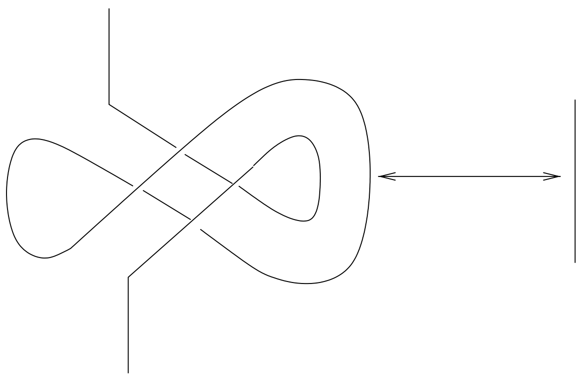

Fig. 12. One Kirby move with one string.

this point, we can see the following transformation of diagram in Figure 10 and Figure 11 which correspond to Figure 8 and Figure 9 respectively.

However, we have diagrams of Kirby moves as in Figure 12 and Figure 13.

If we use Proposition 2·5, for the Kirby move in Figure 12, we have algebraic representation (2·11) as follows,

$$(tr \otimes id)(\Delta(v) \cdot G^{-1} \otimes 1) = \lambda(v)1.$$

We also could see, for Figure 13, that the corresponding algebraic representation is given by (5·1),

$$tr(vG^{-1}) = \lambda(GvG^{-1}) = \lambda(v). \tag{5·1}$$

Generally, for the unknotted component with framing +1, diagrams of Kirby moves is shown in Figure 14. For the Kirby move in Figure 14, corresponding to the formula of Proposition 2·5 as follows

$$(tr \otimes id^{\otimes n})(\Delta^n(v) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v)1^{\otimes n}.$$

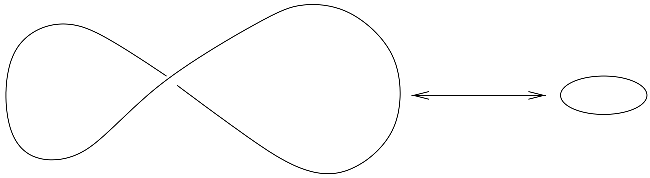

Fig. 13. One Kirby move without string.

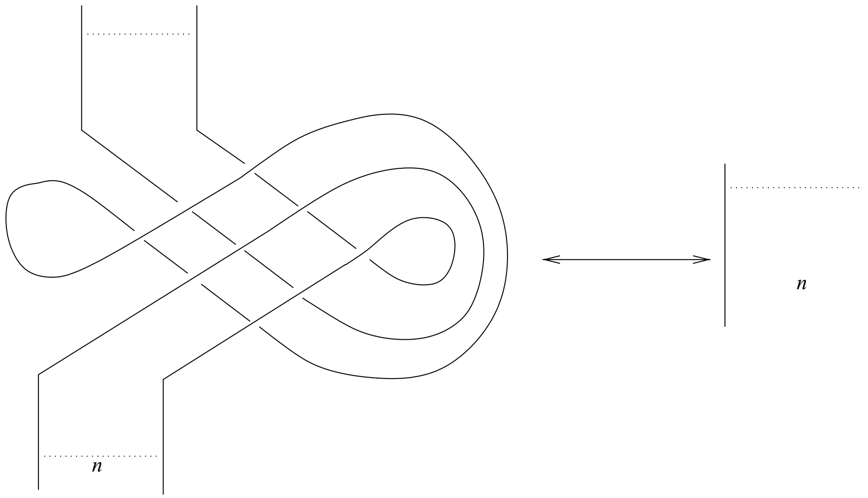

Fig. 14. General Kirby move with n strings.

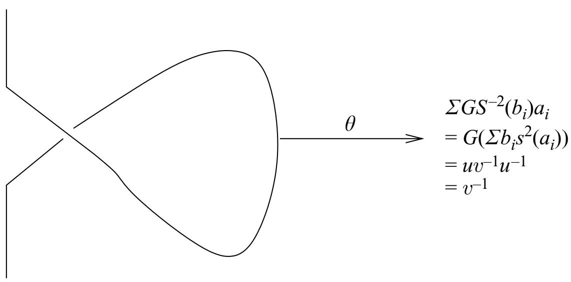

Fig. 15. Algebraic expression for a negative kink.

Similarly to figures in Figures 8, 10 and 11, we can derive the algebraic expression in Figures 15, 16 and 17.

Thus, for the following algebraic formulae (as mentioned in Proposition 2·5):

$$tr(v^{-1}G^{-1} \otimes 1) = \lambda(v^{-1})$$
$$(tr \otimes id)(\Delta(v^{-1}) \cdot G^{-1} \otimes 1) = \lambda(v^{-1})1$$

correspondingly, the diagrams of Kirby moves are in Figures 18 and 19.

For general Kirby moves with the unknotted component with framing -1 in Figure 20, we can represent it by the formula in Proposition 2·5,

$$(tr \otimes id^{\otimes n})(\Delta^n(v^{-1}) \cdot G^{-1} \otimes 1^{\otimes n}) = \lambda(v^{-1})1^{\otimes n}.$$

Therefore we get an algebraic representation of Kirby moves by unimodular finite-dimensional ribbon Hopf algebra. This is a transformation of that in [**9**]. Ohtsuki has also

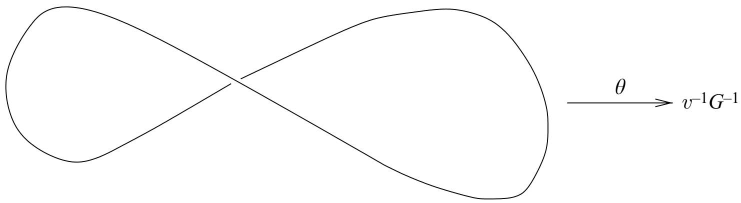

Fig. 16. Algebraic expression for a negative eight.

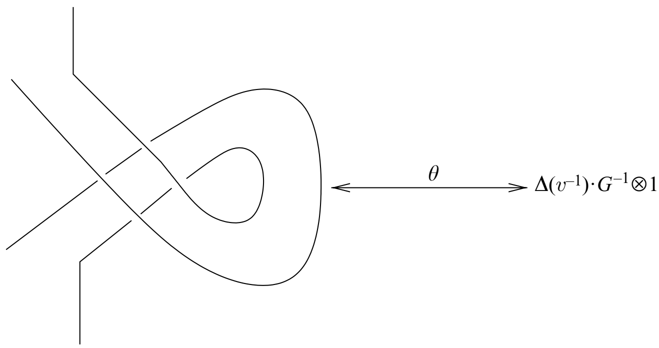

Fig. 17. Algebraic expression for a negative double kink.

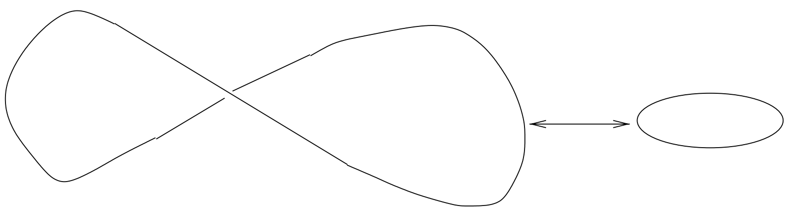

Fig. 18. The simple Kirby move with negative framing.

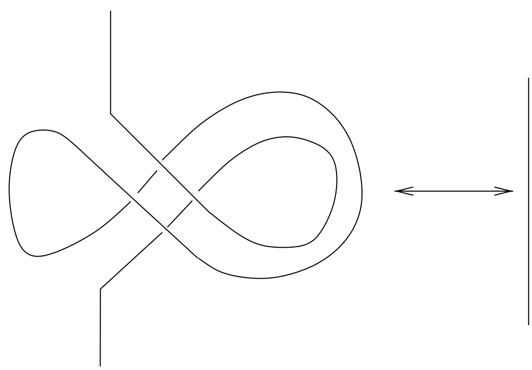

Fig. 19. One Kirby move with negative framing.

given an expression [**5**], which is a transformation of Hennings's [**2**]. But when we use algebra $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ and its accompany $(H, m', \Delta, \mu, \varepsilon, s^{-1}, R^{-1}, v^{-1})$, the results in this section are accompanied with Ohtsuki's [**5**] by the inversion map ρ, and the algebraic representation of Kirby moves in [**9**] is accompanied with that in [**2**] by the inversion map ρ. Essentially, those four kinds of algebraic representation of Kirby moves are

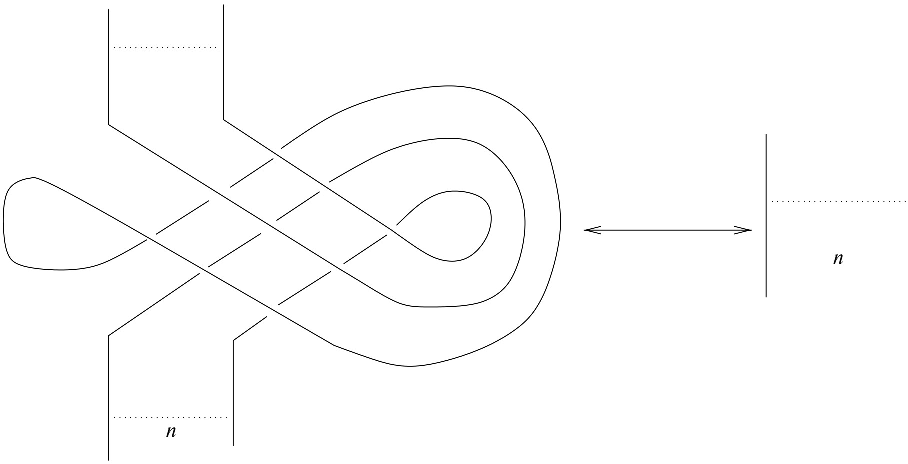

Fig. 20. One Kirby move with n strings and negative framing.

equivalent, when we construct invariants of 3-manifolds. We give the following theorems to state the relations among invariants of 3-manifolds.

THEOREM 5·1. *Given a unimodular finite-dimensional ribbon Hopf algebra, there are only two types of linear maps induced from right integrals, which can be used for the algebraic representation of Kirby moves.*

Proof. Since the dimension of right integral space of a Hopf algebra does not exceed one, and for the given algebra and its company both have one type of linear map induced by its integral respectively, we can only have two types of linear maps.

LEMMA 5·1. *From universal invariants of framed links, 3-manifold invariants defined by Hennings and Kauffman–Raford respectively can be derived*:

$$INK(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} tr^{\otimes c(L)}(\theta(L))$$

$$I_{v,v}(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} v^{\otimes c(L)}(\Phi_{He}(L))$$

where $c(L)$ denotes the number of components of L and $\sigma(L)$ denote the signature of the matrix of linking number of components L.

Proof. By the definition of $\theta(L)$ in [**9**] and $TR(L)$ in [**3**], we can see

$$TR(L) = tr^{\otimes c(L)}(\theta(L))$$

and so,

$$\begin{aligned} INK(L) &= [\lambda(v)\lambda(v^{-1})]\sigma^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} TR(L) \\ &= [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} tr^{\otimes c(L)}(\theta(L)). \end{aligned}$$

By Theorem 3·1 and the structure of $I_{v,v}(L)$ in [**2**], we have

$$[|L|]_{v,v}(L) = v^{\otimes c(L)}(\Phi_{He}(L)).$$

Hence

$$I_{v,\upsilon}(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v^{-1})}{\lambda(v)}\right]^{-\frac{\sigma(L)}{2}} [|L|]_{v,\upsilon}$$
$$= [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v^{-1})}{\lambda(v)}\right]^{-\frac{\sigma(L)}{2}} \upsilon^{\otimes c(L)}(\Phi_{He}(L)).$$

THEOREM 5·2. *Let $(H, m, \Delta, \mu, \varepsilon, s, R, v)$ be a unimodular finite-dimensional ribbon Hopf algebra, $\lambda \in A^*$ be a none-zero right integral and L be a framed link. If $I_{v,\upsilon}(L)$ and $INK(L)$, invariants of the associated 3-manifolds obtained by surgery along L, are respectively defined by Hennings and Kauffman–Radford, then*

$$\overline{\rho}(I_{v,\upsilon}(L)) = INK(L)$$

where $\overline{\rho}$ is the map by the inversion ρ.

Proof. By Lemma 5·1, we have

$$INK(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} tr^{\otimes c(L)}(\theta(L))$$

and

$$I_{v,\upsilon}(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v^{-1})}{\lambda(v)}\right]^{-\frac{\sigma(L)}{2}} \upsilon^{\otimes c(L)}(\Phi_{He}(L)).$$

By proposition 6 in [**2**], and tr in [**3**], we have

$$tr = \lambda \cdot G,$$

so then $\upsilon = tr$.

We already know that $\theta(L) = \rho(\Phi_{He}(L))$. If we take the action of linear map $tr^{\otimes n}$ on the both sides of this equation and the application of ρ_0 and denote the result of ρ action by $\overline{\rho}$, we have

$$INK(L) = \overline{\rho}(I_{v,\upsilon}(L)).$$

THEOREM 5·3.

$$INK(L) = c_\lambda w(M)$$

where $w(M)$ is defined in [**9**], *c_λ is a constant.*

Proof. By $\sigma = \sigma_+ - \sigma_-$, $tr = \varphi = \lambda \cdot G$

$$\frac{INK(L)}{w(M)} = \frac{[\lambda(v)\lambda(v^{-1})]\sigma^{-\frac{c(L)}{2}} \left[\frac{\lambda(v)}{\lambda(v^{-1})}\right]^{-\frac{\sigma(L)}{2}} TR(L)}{\left[\lambda(v^{-1})\right]^{\sigma_+ - c(L)} [\lambda(v)]^{-\sigma_+} \varphi^{\otimes c(L)}(\theta(L))}$$
$$= [\lambda(v)]^{\frac{\sigma_+ + \sigma_- - c(L)}{2}} [\lambda(v^{-1})]^{\frac{c(L) - (\sigma_+ + \sigma_-)}{2}} \frac{tr^{\otimes c(L)}(\theta(L))}{\varphi^{\otimes c(L)}(\theta(L))}$$
$$= (\lambda(v))^{\frac{\sigma_+ - \sigma_- - c(L)}{2}} (\lambda(v^{-1}))^{\frac{c(L) - (\sigma_+ + \sigma_-)}{2}},$$

then

$$INK(L) = (\lambda(v))^{\frac{\sigma_+ - \sigma_- - c(L)}{2}} (\lambda(v^{-1}))^{\frac{c(L) - (\sigma_+ + \sigma_-)}{2}} w(M).$$

For the same 3-manifold obtained from L by surgery, $\sigma_+ - \sigma_- - c(L)$ and $c(L) - (\sigma_+ + \sigma_-)$ are always constants, so we denote $(\lambda(v))^{\frac{\sigma_+ - \ sigma_- - c(L)}{2}}(\lambda(v^{-1}))^{\frac{c(L)-(\sigma_+ + \sigma_-)}{2}}$ by c_λ. Hence, we have

$$INK(L) = c_\lambda w(M).$$

THEOREM 5·4. *If* $\chi = \mu$, *then*

$$I_{v,v}(L) = c_\chi \omega(M),$$

where $\omega(M)$ *is define in* [**5**], c_χ *is a constant.*

Proof. By Ohtsuki [**5**]

$$\omega(M) = c_+^{\sigma_- - c(L)} c_-^{-\sigma_-} \chi^{\otimes c(L)}(\Phi_O(L))$$

and, from 5·1,

$$I_{v,v}(L) = [\lambda(v)\lambda(v^{-1})]^{-\frac{c(L)}{2}} \left[\frac{\lambda(v^{-1})}{\lambda(v)}\right]^{-\frac{\sigma(L)}{2}} v^{\otimes c(L)}(\Phi_{He}(L)).$$

When $\chi = \mu$, it is easy to know that

$$\chi(\lambda v^{-1} \cdot v) = \mu(G \cdot v) = c_-,$$

$$\mu(s(\lambda v^{-1} \cdot v)) = \mu(G^{-1} \cdot v) = \lambda(v),$$

thus

$$c_- = \lambda(v).$$

Similarly, we have

$$c_+ = \lambda(v^{-1}).$$

It is also easy to know that

$$c_\chi = (\lambda(v))^{\frac{\sigma_+ - \sigma_- - c(L)}{2}}(\lambda(v^{-1}))^{\frac{c(L)-(\sigma_+ + \sigma_-)}{2}}$$

is a constant.

Remark 5. By these theorems, we know that invariants of 3-manifold introduced by Hennings, Kauffman–Radford, Ohtsuki and the author, respectively, are equivalent. In order to understand this point, it is a good idea to construct these invariants for a reduced quantum group $(U_q(sl_2))'$ and discuss special properties and relations among them. We hope to discuss this issue elsewhere.

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