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Continuous Time Markov Processes on Graphs

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Abstract: We study continuous time Markov processes on graphs. The notion of frequency is introduced, which serves well as a scaling factor between any Markov time of a continuous time Markov process and that of its jump chain. As an application, we study "multiperson simple random walks" on a graph G with n vertices. There are n persons distributed randomly at the vertices of G. In each step of this discrete time Markov process, a randomly selected person is moved to one of the adjacent vertices selected randomly. We give estimate on the expected number of steps for these n persons to meet all together at a specific vertex, given that they are at different vertices at the begininng. For regular graphs, our estimate is exact.

Keywords: Graphs; Graph frequency; Markov processes; Multiperson simple random walks.

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1. INTRODUCTION

In this article we will consider, for simplicity, connected simple graphs only. These are connected graphs without multiple edges and loops.

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We will adopt the following notations and terminologies for a graph G. The sets of vertices and edges of G are $V(G)$ and $E(G)$, respectively. The order n, of G is the number of vertices of G, and the size m, of G is the number of edges of G. Thus, $n = |V(G)|$ and $m = |E(G)|$. For a vertex $x \in V(G)$, $\Gamma(x)$ is the set of vertices which are connected to x by an edge in $E(G)$. The degree of a vertex x is $d(x) = |\Gamma(x)|$. We have

$$\sum_{x \in V(G)} d(x) = 2m.$$

The adjacent matrix of G, $A(G) = (a_{ij})_{n\times n}$, is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

and the diagonal matrix $D(G)$ has the sequence of degrees of vertices as its corresponding diagonal entries. Finally, we denote

$$d_m = \min\{d(x); x \in V(G)\} \quad \text{and} \quad d_M = \max\{d(x); x \in V(G)\}.$$

For these and other notations and facts on graph theory we refer to [1–3, 6].

What concerns us primarily in this article is the following continuous time Markov process on a graph G: The probability that a person standing at a vertex x of G to jump to a neighboring vertex in $\Gamma(x)$ within a time period Δt is $d(x)\Delta t + o(\Delta t)$, and once jumped, the person at x has equal probability to land at a vertex $y \in \Gamma(x)$. If we write

$$Q = Q(G) = -D(G) + A(G),$$

the transition probability matrix of this Markov process is

$$P(t) = e^{tQ}.$$

We call this Markov process an continuous time simple random walk (CTSRW). In the literature, it is the discrete time simple random walks (SRW) on a graph G that people consider most. One usually call an SRW the jump chain of CTSRW. The transition probability matrix of the jump chain SRW is $D(G)^{-1}A(G)$.

We introduce in this article a fundamental quantity for an CTSRW on a graph G called frequency. Let $N(t)$ be the expected number of jumps of the Markov process CTSRW up to time t. Then the frequency f of an CTSRW is defined by

$$f = \lim_{t\to\infty} \frac{N(t)}{t}.$$

Using the Lévy formula, see Syski [4, 8], we are able to calculate the frequency for CTSRW and get

$$f = \frac{2m}{n}.$$

The frequency turns out to be a natural scaling factor between various important quantities of CTSRW and SRW. For example, we have the following results established infra.

Theorem 1.1. *Let G be nonbipartite. For a vertex x of G, let T_x be the first return time of CTSRW on G and N_{T_x} be the number of jumps during the time period $[0, T_x]$. (Notice that N_{T_x} is the first return time for the jump chain SRW.) Then, the expectations $E(T_x)$ and $E(N_{T_x})$, satisfy the following relation: $E(N_{T_x}) = fE(T_x)$.*

More generally, we have the following theorem.

Theorem 1.2. *Let T be any stopping time of CTSRW on a graph G with finite expectation, let N_T be the number of jumps during the time period $[0, T]$. Then*

$$d_m \leq f \leq d_M,$$

and

$$d_m E(T) \leq E(N_T) \leq d_M E(T).$$

In particular, if G is regular so that $d_m = d_M$, we have $E(N_T) = fE(T)$.

As an application, we consider multiperson simple random walks (MPSRW) on a graph G. It is inspired by coalescent theory in molecular population genetics [5]. To start with, we assume that each of the n vertices of G is occupied by a person. At each step of this Markov chain, there is one person, equally possible for each of these n persons, who moves to a neighboring vertex, also equally possible for each of the neighboring vertices. We would like to know the expected number of steps this Markov chain should take for these n persons to meet all together at a specified vertex.

We will see that this Markov chain is the jump chain of a continuous time Markov process on the set M_n of maps from $\{1, 2, \ldots, n\}$ to itself. On the other hand, this continuous time Markov process on M_n can be identified with the nth tensor power of CTSRW on G. Thus, computation of expectations of various stopping times for this continuous time Markov process on M_n can be carried out. We are then able to use Theorem 1.2 mentioned above to obtain estimates for the expected number of steps for MPSRW on G to coalesce.

We refer the reader to [1, 3, 6, 7] for basic terminologies and results in the study of simple random walks on graphs.

2. CONTINUOUS TIME MARKOV PROCESS ON WEIGHTED GRAPHS

Let G be a connected weighted graph with order n and size m. We define a continuous time Markov process X_t on G by giving its infinitesimal generator Q as the negative weighted Laplacian of G. Specifically, every edge $xy \in E(G)$ is associated with a positive number (weight) w_{xy}. Since the edges in G are not directed, $w_{xy} = w_{yx}$. Define

$$w_x := \sum_{y \in \Gamma(x)} w_{xy}$$

the total weight at the vertex x. We write $Q = (q_{xy})_{n\times n}$, where

$$q_{xy} = \begin{cases} w_{xy} & \text{if } xy \in E(G), \\ -w_x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the probability transition matrix of X_t is given by

$$P(t) = e^{tQ} = (p_{xy}(t))_{n\times n},$$

and transition probability from vertex x to vertex y is given by

$$\Pr\{X(h+t) = y \mid X(h) = x\} = p_{xy}(t).$$

We call $-Q = L_w$ the weighted Laplacian of the weighted graph G.

In the special case of $w_{xy} = 1$ for all $xy \in E(G)$, we have CTSRW on the graph G. The infinitesimal generator $Q = -L(G) = -D(G) + A(G) = (q_{xy})_{n\times n}$ is given by

$$q_{xy} = \begin{cases} 1 & \text{if } xy \in E(G), \\ -d(x) & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

2.1. Ergodicity

Since the infinitesimal generator Q is symmetric, the uniform probability vector $u = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is an invariant measure of the Markov process X_t. That is, we have $uQ = 0$ and

$$uP(t) = u\left(I + tQ + \frac{t^2}{2!}Q^2 + \cdots\right) = u.$$

We claim that u is also the ergodic vector or the stationary distribution. To see this, notice first that the graph G is connected so that the process X_t is irreducible. Thus,

$$\lim_{t\to\infty} p_{xy}(t) =: v_y$$

exists and does not depend on x. Actually, we have $v_y = u_y = \frac{1}{n}$ by the following calculation: First, for any fixed $t > 0$,

$$uP(2t) = uP(t)P(t) = uP(t) = u,$$
$$\cdots$$
$$uP(kt) = uP((k-1)t) = \cdots = uP(t) = u.$$

Next

$$u_y = \sum_x u_x p_{xy}(kt), \quad \text{for } y \in V(G).$$

Finally, let $k \to \infty$ to get

$$u_y = \sum_x u_x v_y = v_y.$$

2.2. The Mean First Return Time

Corresponding to a vertex $x \in V(G)$, denote by T_{xx} the first return time to x, given that the Markov process X_t starts at x. That is,

$$T_{xx} = \inf\{t : t > \rho_x, X_t = x \mid X_0 = x\}$$

where ρ_x is the exit time from the vertex x. We denote by $h(x, x)$ the mean first return time $E(T_{xx})$.

Lemma 2.1. *The mean first return time is given by*

$$h(x, x) = \frac{n}{w_x}.$$

Proof. Write

$$F_{xx}(t) = \Pr\{T_{xx} \le t\}.$$

Then we have the equation, see Chung [4],

$$p_{xx} = e^{-w_x t} + \int_0^t p_{xx}(t-s)dF_{xx}(s).$$

Taking the Laplace transform, we get

$$\begin{aligned}
\phi_{xx}(\lambda) &= \int_0^\infty e^{-\lambda t} p_{xx}(t)dt \\
&= \int_0^\infty e^{-(\lambda+w_x)t}dt + \int_0^\infty e^{-\lambda t}\int_0^t p_{xx}(t-s)dF_{xx}(s)dt \\
&= \frac{1}{\lambda+w_x} + \int_0^\infty \int_s^\infty e^{-\lambda t} p_{xx}(t-s)dt\, dF_{xx}(s) \\
&= \frac{1}{\lambda+w_x} + \int_0^\infty \int_0^\infty e^{-\lambda(v+s)} p_{xx}(v)dv\, dF_{xx}(s) \\
&= \frac{1}{\lambda+w_x} + \int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\lambda v} p_{xx}(v)dv\, dF_{xx}(s) \\
&= \frac{1}{\lambda+w_x} + \int_0^\infty e^{-\lambda s}\phi_{xx}(\lambda)dF_{xx}(s) \\
&= \frac{1}{\lambda+w_x} + \phi_{xx}(\lambda)l_x(\lambda),
\end{aligned}$$

where $l_x(\lambda) = \int_0^\infty e^{-\lambda s}dF_{xx}(s)$. Then, we have

$$\lambda\phi_{xx}(\lambda) = \frac{1}{\lambda+w_x}\left(\frac{1-l_x(\lambda)}{\lambda}\right)^{-1}.$$

Since

$$\lim_{\lambda\to 0}\frac{1-l_x(\lambda)}{\lambda} = \lim_{\lambda\to 0}\int_0^\infty se^{-\lambda s}dF_{xx}(s) = \int_0^\infty sdF_{xx}(s) = h(x, x)$$

and

$$\lim_{\lambda\to 0}\lambda\phi_{xx}(\lambda) = u_x,$$

we get

$$\begin{aligned}
u_x &= \lim_{\lambda\to 0}\frac{1}{\lambda+w_x}\left(\frac{1-l_x(\lambda)}{\lambda}\right)^{-1} \\
&= \frac{1}{h(x, x)w_x}.
\end{aligned}$$

By ergodicity, $u_x = \frac{1}{n}$ and we have $h(x, x) = \frac{n}{w_x}$. □

2.3. The Mean Hitting Time of y from x

In general, define

$$T_{xy} = \inf\{t : t > \rho_x, X_t = y \mid X_0 = x\},$$

that is, T_{xy} is the time of first entrance into, or hitting, the vertex y, given that the process starts at x. We denote this mean first hitting time of y from x by $h(x, y) = E(T_{xy})$. We have the following equation:

$$h(x, x) = \frac{1}{w_x} + \sum_{y\in\Gamma(x)} \frac{w_{xy}}{w_x} h(y, x) = \frac{1}{w_x} + \frac{1}{w_x} \sum_{y\in\Gamma(x)} w_{xy} h(y, x).$$

Lemma 2.2. *The mean hitting time is provided by*:

$$h(x, y) = n \int_0^\infty (p_{yy}(t) - p_{xy}(t))dt.$$

The formula in this lemma is similar to the formula in the discrete time case. We omit the proof since it is also analogous to the discrete time case.

2.4. The Stationary Distribution of SRW

For an unweighted graph G, the jump chain of CTSRW on G is the SRW on G. We know that $Q = -D + A$ and that the transition probability matrix of SRW is $D^{-1}A$. If we set $\tilde{\pi} = uD$, where u is the stationary distribution of CTSRW, then $uQ = -uD + uA = 0$. So, substitute $u = \tilde{\pi}D^{-1}$ we have

$$\tilde{\pi} = \tilde{\pi}D^{-1}A.$$

Thus, $\tilde{\pi}$ is an invariant measure of SRW. We need to normalize it. Let

$$\pi = \frac{1}{\sum_{i=1}^{n} \tilde{\pi}_i} \tilde{\pi}.$$

Then π is an invariant distribution for SRW. Specifically

$$\pi_x = \frac{d(x)}{\sum_{x\in V(G)} d(x)} = \frac{d(x)}{2m}.$$

If the graph G is nonbipartite, this invariant distribution is also the stationary distribution.

It is well known that the mean number of steps SRW should take to return to the vertex x for the first time is $1/\pi_x = 2m/d(x)$. Recall that the mean first return time of CTSRW is $n/d(x)$. Therefore, it is natural to think of the quantity

$$f = \frac{2m/d(x)}{n/d(x)} = \frac{2m}{n}$$

as the frequency (number of jumps per unit time) of CTSRW. We will make this notion precise in the following subsection.

2.5. The Frequency

To define the frequency for the continuous time Markov process X_t on a weighted graph, we first define a quantity $N(t)$ for $t > 0$:

$$N(t) = E \text{ (the number of jumps of } X_t \text{ up to time } t).$$

Theorem 2.1. *We have*

$$f := \lim_{t\to\infty} \frac{N(t)}{t} = \frac{2w}{n}$$

where w is the total weight of G given by

$$w = \frac{1}{2} \sum_{xy\in E(G)} w_{xy}.$$

Proof. Let us recall the Lévy formula first, see Chung [4]. Given a Markov process X_t, we consider a purely discontinuous functional $A = \{A_t : 0 < t < \infty\}$ on the path space defined by

$$A_t := \sum_{0<s\le t} g(X_{s^-}, X_s), \quad t > 0,$$

where g is a function on $V(G) \times V(G)$. Also, we define a function b_Q on $V(G)$ by

$$b_Q(x) := \sum_{y\neq x} w_{xy} g(x, y), \quad x \in V(G),$$

and the integral functional $B = \{B_t : 0 \le t < \infty\}$ on the path space is defined by

$$B_t = \int_0^t b_Q(X_s)ds = \int_0^t \sum_{y\neq X_s} w_{X_s,y} g(X_s, y)ds.$$

Then, the relationship between the functionals A and B is given by the Lévy formula:

$$E_x \sum_{0<s\leq t} g(X_{s^-}, X_s)\alpha(t) = E_x \int_0^t \alpha(t) b_Q(X_s) ds, \quad t > 0,$$

for any continuous positive function $\alpha(t)$.

Now, taking $\alpha(t) = 1$, the Lévy formula tells us

$$E_x A_t = E_x B_t = \int_0^t P(s) \cdot b_Q(x) ds.$$

Furthermore, let

$$g(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then A_t is the number of transitions of states of X_t up to time t, that is, $EA_t = N(t)$.

We start at the vertex x. Then

$$E_x A_t = \int_0^t P(s) \cdot b_Q(x) ds = \int_0^t \sum_{y\in V(G)} p_{xy}(s) w_y \, ds.$$

If we start at an initial distribution θ on graph G, then

$$E_\theta A_t = \int_0^t \theta P(s) \cdot b_Q ds = \int_0^t \sum_{x,y} \theta_x p_{xy}(s) w_y \, ds.$$

Thus, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{E_\theta A_t}{t} &= \lim_{t\to\infty} \frac{\int_0^t \sum_{x,y} \theta_x p_{xy}(s) w_y \, ds}{t} \\ &= \lim_{t\to\infty} \sum_{x,y} \theta_x p_{xy}(t) w_y \\ &= \sum_{x,y} \theta_x u_y w_y = \sum_y u_y w_y \end{aligned}$$

and this is independent of the initial condition θ. So,

$$f = \sum_{y\in V(G)} \frac{1}{n} w_y = \frac{2w}{n}.$$

□

Using the notion of frequency, we can compare various Markov times for the continuous time Markov process and its jump chain. Let us recall the notion of Markov time (or stopping time) first. Associated with a stochastic process, there are random variables independent of the future. This kind of random variables are called Markov time or stopping time. Specifically, let σ be a nonnegative random variable associated with a given process $\{X_t : 0 \leq t \leq \infty\}$. Here, σ associates with each sample function X_t a nonnegative number which we denote by $\sigma(X_t)$. Such a random variable σ is said to be a Markov time relative to the process X_t if it has the following property:

If X_t and Y_t are two sample functions of the process such that $X_\tau = Y_\tau$ for $0 \leq \tau \leq s$ and $\sigma(X_t) < s$, then $\sigma(X_t) = \sigma(Y_t)$.

We shall now establish our main result.

Theorem 2.2 (Time-Step Inequality). *Let T be any stopping time associated with the continuous Markov process X_t on a weighted graph G and with $E(T) < \infty$. Define $w_m = \min\{w_x; x \in V(G)\}$ and $w_M = \max\{w_x; x \in V(G)\}$, and let N_T be the number of jumps of X_t during the period $[0, T]$. Then, we have*

$$w_m \leq f \leq w_M, \quad \text{and} \quad w_m E(T) \leq E(N_T) \leq w_M E(T).$$

Proof. For an initial distribution θ, we have from the Lévy formula that

$$N(t) = E_\theta A_t = \int_0^t \sum_{x,y \in V(G)} \theta_x p_{xy}(s) w_y \, ds.$$

Since

$$\sum_{x,y \in V(G)} \theta_x p_{xy}(s) w_y \leq \sum_{x,y \in V(G)} \theta_x p_{xy}(s) w_M = w_M$$

and

$$\sum_{x,y \in V(G)} \theta_x p_{xy}(s) w_y \geq \sum_{x,y \in V(G)} \theta_x p_{xy}(s) w_m = w_m,$$

we get

$$w_m t \leq N(t) \leq w_M t.$$

This is

$$w_m \leq \frac{N(t)}{t} \leq w_M.$$

By taking limit, we have

$$w_m \leq f \leq w_M.$$

Let $F(t) := Pr\{T < t\}$ denote the distribution of the stopping time T. Then we have

$$\int_0^\infty w_m t\, dF \leq \int_0^\infty \frac{N(t)}{t} t\, dF \leq \int_0^\infty w_M t\, dF.$$

Since

$$E(N_T) = E(E(A_T \mid T = t)) = \int_0^\infty N(t) dF,$$

we get

$$w_m E(T) \leq E(N_T) \leq w_M E(T). \qquad \square$$

The following are two interesting corollaries. The proofs of them are obvious, so we just state the results.

Corollary 2.1 (Time-Step Inequality). *For CTSRW on a graph G and any Markov time T with finite expectation, we have*

$$d_m \leq f \leq d_M \quad \text{and} \quad d_m E(T) \leq E(N_T) \leq d_M E(T),$$

where d_m and d_M are defined similar to w_m and w_M, respectively.

Corollary 2.2. *If G is a regular graph with constant degree d at each vertex, then $f = d$ and $E(N_T) = fE(T)$, for any stopping time T associated with CTSRW on G.*

We shall call the inequalities in Theorem 2.2 and Corollary 2.1 "time-step inequalities." Of course, we have another version given by:

$$\frac{E(N_T)}{d_M} \leq E(T) \leq \frac{E(N_T)}{d_m}.$$

In a sense, those inequalities characterize the timing difference between CTSRW and SRW on a graph. It is also interesting to see that the frequency of an unweighted graph is the average of the eigenvalues of its Laplacian. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ be the spectrum of the Laplacian $L(G)$ of a graph G, $L(G) = D(G) - A(G)$. Then, $\sum_{i=1}^n \lambda_i = \sum_{x \in V(G)} d(x) = 2m$. Thus,

$$f = \frac{\sum_{i=1}^n \lambda_i}{n}.$$

Theorem 2.3. *Let G be a nonbipartite graph, and T_x be the first return time of CTSRW on G. Then $E(N_{T_x}) = fE(T_x)$.*

Proof. By the theory of discrete time simple random walks on a graph G, we know $E(N_{T_x}) = \frac{2m}{d(x)}$, see [6].

For CTSRW on G, we know $E(T_x) = \frac{n}{d(x)}$, and also $f = \frac{2m}{n}$. Thus,

$$E(N_{T_x}) = \frac{2m}{d(x)} = \frac{n}{d(x)} \cdot \frac{2m}{n} = fE(T_x). \qquad \square$$

Now, we consider a special problem as that in SRW. Let G be a connected nonbipartite graph. We start our CTSRW at a vertex x and fix a neighboring vertex y of x. What is the expected time that our CTSRW should take in order to return to x through the edge yx?

For an SRW on G, we know that the expected number of steps one should take in order to return to x through the edge yx, is $2m$, see Bollobás [1]. To deal with the problem for CTSRW on G, we formulate the following Markov time:

$$T_{(x,\overline{yx})} := \inf\{t + \rho_y : t > 0, X(s+t) = y, X(s+t+\rho_y) = x \mid X(s) = x\}.$$

This is the fixed edge first return time, then $E(N_{T_{(x,\overline{yx})}}) = 2m$. By our time-step inequality, we have the following corollary.

Corollary 2.3. *The mean fixed edge first return time has bounds as*

$$\frac{2m}{d_M} \le E(T_{(x,\overline{yx})}) \le \frac{2m}{d_m}.$$

The following is another case where the frequency gives us a perfect scaling factor between corresponding quantities of CTSRW and SRW.

Lemma 2.3. *Let G be a connected graph of order n and size m. The mean hitting time $h(x, y)$ of the CTSRW on G satisfies*

$$\sum_{x\in V(G)} \sum_{y\in\Gamma(x)} h(y, x) = n(n-1).$$

Proof. We know that

$$h(x, x) = \frac{1}{d(x)} + \frac{1}{d(x)} \sum_{y\in\Gamma(x)} h(y, x) = \frac{n}{d(x)}.$$

So, we have $\sum_{y\in\Gamma(x)} h(y, x) = n - 1$ which is independent of x. Thus, for n vertices, we will have

$$\sum_{x\in V(G)} \sum_{y\in\Gamma(x)} h(y, x) = n(n-1). \qquad \square$$

If we denote the hitting time of y from x in SRW by $H(x, y)$, then we have the equality

$$\sum_{x \in V(G)} \sum_{y \in \Gamma(x)} H(y, x) = 2m(n-1),$$

see Bollobás [1]. So, we have the following theorem.

Theorem 2.4. *With notations as the above, we have*

$$\sum_{x \in V(G)} \sum_{y \in \Gamma(x)} H(y, x) = f \sum_{x \in V(G)} \sum_{y \in \Gamma(x)} h(y, x).$$

Also, for CTSRW on a graph G, we define the mean commute time between vertices x and y to be $c(x, y) = h(x, y) + h(y, x)$. Let $C(x, y)$ be the corresponding quantity for SRW on G. Then we have another version of the above equation in Theorem 2.4 as

$$\sum_{xy \in E(G)} C(x, y) = f \sum_{xy \in E(G)} c(x, y).$$

3. MULTIPERSON SIMPLE RANDOM WALKS ON GRAPHS

We are led to the MPSRW on a graph G by the study of a continuous time Markov process induced by CTSRW on G. The combinatorics of MPSRW is much richer than we have touched upon here.

Let I_n be a finite set of cardinality n. For example, we may have $I_n = \{1, 2, \dots, n\}$. We denote the set of all maps from I_n to itself by M_n. We have the symmetric group S_n sitting inside of M_n. A map $x \in M_n$ is called a generalized permutation of deficiency k if

$$|x(I_n)| = n - k.$$

We denote this by $\operatorname{def}(x) = k$.

Note that M_n is a semigroup under composition. The symmetrical group S_n is a subgroup of M_n. The deficiency determines a grading on M_n which is compatible with the semigroup product on M_n :

$$\operatorname{def}(x) + \operatorname{def}(y) \geq \operatorname{def}(x \circ y).$$

Define $M_n^{(k)} := \{\text{all maps with deficiency } k\}$. Then we have a decomposition of M_n according to the deficiency:

$$M_n = M_n^{(0)} \amalg M_n^{(1)} \amalg M_n^{(2)} \amalg \cdots \amalg M_n^{(n-1)},$$

where $M_n^{(0)} = S_n$, and $\amalg$ is disjoint union.

Let G be a graph with the set of vertices $V(G)$ identified with I_n. We will call G the ground graph. We define a new graph $M(G)$ as follows: The set of vertices of $M(G)$ is M_n; For $x, y \in M_n$, there is an edge xy in $M(G)$ only when

$$|\{i : x(i) \neq y(i)\}| = 1,$$

and if $x(i) \neq y(i)$, then $a_{x(i)y(i)} = 1$. We will see that there is a close relationship between the graph $M(G)$ and the nth tensor power of CTSRW on the ground graph G.

3.1. The Tensor Product of Markov Processes

Let $X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)}$ be Markov processes on the state spaces $S^{(1)}, S^{(2)}, \ldots, S^{(n)}$ respectively. We define a new process Y_t on the state space $S^{(1)} \times S^{(2)} \times \cdots \times S^{(n)}$ with the transition probability given by

$$\begin{aligned}\Pr\{Y_{t+h} = (s_2) \mid Y_h = (s_1)\} &= \prod_{k=1}^{n} \Pr\left\{Y_{t+h}^{(k)} = s_2^{(k)} \mid Y_h^{(k)} = s_1^{(k)}\right\} \\ &= \prod_{k=1}^{n} p^{(k)}_{s_1^{(k)} s_2^{(k)}}(t),\end{aligned}$$

where $(s_i) = (s_i^{(1)}, s_i^{(2)}, \ldots, s_i^{(n)})$, $i = 1, 2$, and $p^{(k)}_{s_1^{(k)} s_2^{(k)}}(t)$ is the transition probability of the Markov process $X_t^{(k)}$, $k = 1, 2, \ldots, n$. We call Y_t the tensor product of Markov processes $X_t^{(k)}$, $k = 1, 2, \ldots, n$. The next two lemmas can be proved by some direct computations. So we omit the proofs.

Lemma 3.1. *Y_t is a Markov process.*

Lemma 3.2. *Let $q_{(s_1)(s_2)}$ be the infinitesmal generator of Y_t. Then*

$$q_{(s_1)(s_2)} = \begin{cases} q^{(k)}_{s_1^{(k)} s_2^{(k)}} & \text{if } \exists \text{ only one index } k \text{ such that } s_1^{(k)} \neq s_2^{(k)}, \\ \sum_{k=1}^{n} q^{(k)}_{s_1^{(k)} s_2^{(k)}} & \text{if } (s_1) = (s_2), \text{ namely, } s_1^{(k)} = s_2^{(k)} \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

where $q^{(k)}_{s_1^{(k)} s_2^{(k)}}$ is the infinitesmal generator of Markov process $X_t^{(k)}$, $k = 1, 2, \ldots, n$.

When we order the elements of the state space $S^{(1)} \times S^{(2)} \times \cdots \times S^{(n)}$ lexicographically, it is easy to see that the probability transition matrix of Y_t is given by the tensor product

$$P^{(1)}(t) \otimes P^{(2)}(t) \otimes \cdots \otimes P^{(n)}(t).$$

This is why we call Y_t as the tensor product of Markov process $X^{(k)}(t)$, $k = 1, 2, \ldots, n$. By the Lemma 3.2 above, we also can see that the infinitesimal generator matrix of Y_t is given by

$$Q^{(1)} \otimes I \otimes \cdots \otimes I + I \otimes Q^{(2)} \otimes I \otimes \cdots \otimes I + \cdots\cdots + I \otimes \cdots \otimes I \otimes Q^{(n)},$$

where $Q^{(k)}$ is the infinitesimal generator matrix of $X_t^{(k)}$, $k = 1, 2, \ldots n$. Thus, for convenience, we denote $Y_t = X_t^{(1)} \otimes X_t^{(2)} \otimes \cdots \otimes X_t^{(n)}$.

3.2. CTSRW on $M(G)$

On the graph $M(G)$, we have the Markov process CTSRW denoted by Y_t. Let X_t be the Markov process CTSRW on G.

Lemma 3.3. *The Markov process Y_t on the graph $M(G)$ is the nth tensor power of the Markov process X_t on the ground graph G. The jump chain of Y_t is the MPSRW of G.*

Proof. Recall that $Q = -L(G) = -D(G) + A(G) = (q_{ij})_{n\times n}$,

$$q_{ij} = \begin{cases} 1 & \text{if } ij \in E(G), \\ -d(i) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $d(i)$ is the degree of the vertex i. Let $X_t \otimes X_t \otimes \cdots \otimes X_t = Z_t$ be the nth tensor power of the process X_t. Take two states for Z_t, (s_1) and (s_2). Then (s_i) actually is a sequence of vertices of G. We write $(s_1) = (i_1, i_2, \ldots, i_n)$ and $(s_2) = (j_1, j_2, \ldots, j_n)$. By Lemma 3.2, we have

$$p'_{(s_1)(s_2)}(0) = \begin{cases} q^{(k)}_{s_1^{(k)} s_2^{(k)}} & \text{if } \exists \text{ only one index } k \text{ such that } s_1^{(k)} \neq s_2^{(k)}, \\ \sum_{k=1}^{n} q^{(k)}_{s_1^{(k)} s_2^{(k)}} & \text{if } (s_1) = (s_2), \text{ namely, } s_1^{(k)} = s_2^{(k)} \text{ for all } k, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \exists \text{ only one index } k \text{ such that} \\ & \quad i_k \neq j_k,\ i_k j_k \in E(G), \\ -\sum_{k=1}^{n} d(i_k) & \text{if } i_k = j_k \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can identify (s_1) and (s_2) with the images of certain maps x and y, respectively. Then by the definition of $M(G)$, $xy \in E(M(G))$, if and only if $p'_{(s_1)(s_2)}(0) = 1$. As to the degree of x, we know that the neighbors of $x = (s_1)$ can only be $(k_1, i_2, \ldots, i_n)$, where k_1 must be a neighbor of i_1; $(i_1, k_2, \ldots, i_n)$, where k_2 must be a neighbor of i_2; and so on, up to the last one $(i_1, i_2, \ldots, i_{n-1}, k_n)$, where k_n must be a neighbor of i_n. Thus,

$$d(x) = d(s_1) = \sum_{j=1}^{n} d(i_j).$$

This agrees with $p'_{(s_1)(s_1)}(0) = -\sum_{k=1}^{n} d(i_k)$. So, $Y_t = X_t \otimes X_t \otimes \cdots \otimes X_t = Z_t$.

The second conclusion is easy to see. □

The Lemma 3.3 suggests that we may call graph $M(G)$ the tensor power of the graph G. We also note that the transition probability from x to y in M_n is given by

$$P_{xy}(t) = \prod_{i=1}^{n} p_{x(i)y(i)}(t).$$

From the proof of Lemma 3.3, we know that for $x \in M_n = V(M(G))$,

$$d(x) = \sum_{k=1}^{n} d(x(k)).$$

We can compute the size of $M(G)$, that is the number of edges of $M(G)$ as

$$\begin{aligned} 2|E(M(G))| &= \sum_{x \in M_n} d(x) = \sum_{x \in M_n} \sum_{k=1}^{n} d(x(k)) = \sum_{k=1}^{n} \sum_{x \in M_n} d(x(k)) \\ &= \sum_{k=1}^{n} ((d(1) + d(2) + \cdots + d(n))n^{n-1}) \\ &= 2mn^n. \end{aligned}$$

Thus, we have the following corollary.

Corollary 3.1. *The size of $M(G)$ is mn^n, and the order of $M(G)$ is n^n. Therefore, the frequency of $M(G)$ is $2m$.*

Lemma 3.4. *G is bipartite if and only if $M(G)$ is bipartite.*

Proof. We use the classical result of König that a graph is bipartite if and only if all its cycles are even.

If G is not bipartite, then there is a cycle $C = v_1 v_2 \dots v_m v_1$ of odd length in G. We look at the cycle in $M(G)$ given by

$$(v_1, v_1, \dots, v_1)(v_1, \dots, v_1, v_2)(v_1, \dots, v_1, v_3) \\ \cdots (v_1, \dots, v_1, v_m)(v_1, \dots, v_1, v_1).$$

Its length is also odd. Thus, $M(G)$ is not bipartite.

If G is bipartite, then the set of vertices $V = V(G)$ can be written as $V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$ and there is no edge between vertices both in V_1 or both in V_2. We try to bipart the set of vertices of $M(G)$. We know $V(M(G)) = V^{\times n}$. We write $V = V_1 + V_2$. Then

$$V^{\times n} = (V_1 \cup V_2)^{\times n} = V_1^n \cup V_1^{n-1} V_2 \cup V_1^{n-2} V_2^2 \cup \cdots \cup V_1 V_2^{n-1} \cup V_2^n,$$

where $V_1^{n-k} V_2^k$ means we take $(n-k)$ vertices from V_1 and k vertices from V_2, regardless of order, to form a vertex of $M(G)$. Let

$$\overline{V_1} = V_1^n \cup V_1^{n-2} V_2^2 \cup V_1^{n-4} V_2^4 \cup \dots,$$
$$\overline{V_2} = V_1^{n-1} V_2 \cup V_1^{n-3} V_2^3 \cup V_1^5 V_2^{n-5} \cup \dots.$$

Then $\overline{V_1} \cup \overline{V_2} = V(M(G))$ and $\overline{V_1} \cap \overline{V_2} = \emptyset$. By the definition of $M(G)$, we cannot find an edge between any two vertices which are both in $\overline{V_1}$ or both in $\overline{V_2}$. □

If G is not bipartite with order n and size m, write the degree sequence of G as $d_1 \le d_2 \le \cdots \le d_n$. We have the following two corollaries. Because they are easy to derive from Corollary 3.1 and Theorem 2.3, we will not give their proofs.

Corollary 3.2. *For any vertex x in $M(G)$, the x-component of the ergodic vector for SRW on $M(G)$ is given by*

$$\pi_x = \frac{d(x)}{2mn^n} = \frac{\sum_{i=1}^n d(x(i))}{2mn^n} = n^{-n} \sum_{i=1}^n \frac{d(x(i))}{2m} = n^{-n} \sum_{i=1}^n \pi_{x(i)}^G,$$

where

$$\pi_{x(i)}^G = \frac{d(x(i))}{2m}$$

is the $x(i)$-component of the ergodic vector for SRW on G. The expected first return time for CTSRW on $M(G)$ is given by

$$h(x, x) = \frac{n^n}{d(x)} = \frac{n^n}{\sum_{k=1}^n d(x(k))}.$$

In particular, for the identity map Id, $d(Id) = d_1 + d_2 + \cdots + d_n = 2m$. So

$$h(Id, Id) = \frac{n^n}{2m}.$$

Corollary 3.3. *For the jump chain SRW on $M(G)$, or MPSRW on G, we have*

$$H(x,x) = f_{M(G)} h(x,x) = \frac{2mn^n}{\sum_{k=1}^n d(x(k))},$$

where $f_{M(G)} = 2m$ is the frequency of $M(G)$. In particular,

$$H(Id, Id) = n^n.$$

3.3. Hitting Time for CTSRW on $M(G)$

For CTSRW on $M(G)$, we are interested in calculating the mean hitting time $h(Id, c_i)$, where c_i is the constant map to a vertex i of G. Once we know $h(Id, i)$, we can use it to estimate $H(Id, c_i)$ for SRW on $M(G)$, or equivalently, the expect number of steps MPSRW on G should take to have all persons meet at the vertex i. Namely, we have

$$nd_1 h(Id, c_i) \leq H(Id, c_i) \leq nd_n h(Id, c_i).$$

Notice that when the graph G is regular, $H(Id, c_i)$ is determined completely by $h(Id, c_i) : H(Id, c_i) = nd \cdot h(Id, c_i)$.

Let x and y be two distinct vertices in $M(G)$, we consider the hitting time $h(x, y)$ of y from x in CTSRW. We have

$$\begin{aligned} h(x,y) &= n^n \int_0^\infty (P_{yy}(t) - P_{xy}(t))dt \\ &= n^n \int_0^\infty \left(\prod_{i=1}^n p_{y(i)y(i)}(t) - \prod_{i=1}^n p_{x(i)y(i)}(t) \right) dt. \end{aligned}$$

For $x = Id$, $y = c_i$, we have

$$h(Id, c_i) = n^n \int_0^\infty \left(p_{ii}(t)^n - \prod_{j=1}^n p_{ji}(t) \right) dt.$$

To calculate this integral, we diagonalize the Laplacian of G. Write

$$U^T QU = \text{diag}[-\lambda_1, -\lambda_2, \ldots, -\lambda_n],$$

where $U = (u_{ij})_{n\times n}$ is an orthogonal matrix. Then

$$p_{ij}(t) = \sum_{k=1}^n u_{ik} u_{jk} e^{-\lambda_k t}.$$

Since

$$p_{ii}(t)^n = \left(\sum_{k=1}^{n} u_{ik}^2 e^{-\lambda_k t} \right)^n = \sum_{1 \le k_1, k_2, \ldots, k_n \le n} u_{ik_1}^2 u_{ik_2}^2 \cdots u_{ik_n}^2 e^{-(\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n})t}$$

and

$$\prod_{k=1}^{n} p_{ki}(t) = \sum_{1 \le k_1, k_2, \ldots, k_n \le n} u_{1k_1} u_{ik_1} u_{2k_2} u_{ik_2} \cdots u_{nk_n} u_{ik_n} e^{-(\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n})t},$$

we get

$$h(Id, c_i) = n^n \sum_{1 \le k_1, k_2, \ldots, k_n \le n} \frac{u_{ik_1}^2 u_{ik_2}^2 \cdots u_{ik_n}^2 - u_{1k_1} u_{ik_1} u_{2k_2} u_{ik_2} \cdots u_{nk_n} u_{ik_n}}{\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n}}.$$

We conclude this work with an illustrative example.

Example. Let G be the triangle graph. It is regular with $n = 3$ and $d = 2$. The matrix Q and U are given below:

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

We have $U^T Q U = \text{diag}[-3, -3, 0]$. Since $\lambda_3 = 0$, when calculating $h(Id, c_i)$, we should drop the term $(k_1, k_2, k_3) = (3, 3, 3)$. Since the numerator of this term in $h(Id, c_i)$ is also zero, we can drop this term. We, thus, have

$$h(Id, c_1) = 3^3 \cdot \frac{31}{162} = \frac{31}{6}.$$

So

$$H(Id, c_1) = 3 \cdot 2 \cdot \frac{31}{6} = 31.$$

Thus, for our MPSRW on the triangle graph, it takes 31 steps on average for 3 persons to meet at any specified vertex, given that they all start at different vertices.

For the square graph with $n = 4$ and $d = 2$, a similar calculation shows

$$h(Id, c_1) = 4^4 \cdot \frac{167}{1120} = \frac{1336}{35}$$

and

$$H(Id, c_1) = 8 \cdot \frac{1336}{35} \approx 305.371.$$

Thus, for our MPSRW on the square graph, it takes about 305 steps on average for 4 persons to meet at any specified vertex, given that they all start at different vertices.

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