# Subexponential estimates for the first hitting time of a Brownian motion with singular drift

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We study the effect of a power law drift on the Brownian motion in the positive half-line, where the order of the drift at 0 and $\infty$ is different. The first hitting time of 0 is finite almost surely, even when the drift near 0 is positive and unbounded. We obtain subexponential estimates for the tail distribution of the hitting time of 0, that are independent of the singular behavior of the drift near the origin.

*Keywords:* Brownian motion; Brownian motion with drift; half-line; h-transform; large deviations; stochastic comparison

## 1. Introduction

For a real-valued process $X_t$, let $\tau_R(X) = \inf\{t > 0 : X_t = R\}$ denote the first time the process hits the level $R$. The first hitting time of a stochastic process has been widely studied due to its significant applications in both theoretical and applied probability. These applications span various fields, including finance and chemical physics, where models are often driven by physical or economic considerations.

In finance, a common example involves analyzing interest rate fluctuations to make decisions for options based on when rates reach certain thresholds. The Cox-Ingersoll-Ross (CIR) family of diffusions in interest theory, solve $\mathrm{d}X_t = (a + bX_t)\,\mathrm{d}t + c\sqrt{|X_t|}\mathrm{d}B_t$. They are shown in [14] to be a transformation of a Bessel process, so that the threshold for the CIR process is reduced to the more tractable Bessel diffusion.

In chemical physics, the dissociation of a molecule might occur when a system reaches a critical level. These phenomena are mathematically described by various stochastic differential equations, such as those of perturbed Brownian motion or Bessel processes and their generalizations.

Key questions naturally arise, such as under what conditions a critical point is reached, when it will occur, and how the process behaves once the threshold is attained. Some probabilistic answers to these questions are provided by the probability density of the first hitting time, which can sometimes be explicitly calculated. Notable examples include the classical Brownian motion with drift as discussed in [18] and [4], the Ornstein-Uhlenbeck process ([1]), and certain Bessel bridges ([15]). Additionally, an integral formula for the hitting time density of a geometric Brownian motion is presented in [8], which also facilitates sharp estimates for the hitting times of Bessel processes ([7]). These results are often derived using transformations, such as the Lamperti transformation ([20]), that relate Bessel processes to geometric Brownian motion. Series expansions for hitting time densities of some diffusion processes, including the first two moments for Bessel processes, are given in [19] and are linked to the eigenvalues of the associated infinitesimal generator. Further discussions on exponential integral functionals for Brownian motion with drift and Bessel processes are found in [6].

Despite the variety of techniques developed in these studies, exact expressions for the first hitting time densities remain unknown in general. To address this, various numerical methods have been explored, including Monte Carlo simulations for general one-dimensional diffusions ([16]) and weak approximations using Euler schemes for multidimensional diffusions ([13]), among others. The study of large values of the first passage time also has a long history, with significant contributions such as those by [21] and [3] for random walks, [2] for non-stationary Ornstein-Uhlenbeck processes with statistical applications, [11] for Bessel and squared radial Ornstein-Uhlenbeck processes, [24] for the occupation time of Brownian motion integrals, and [10] for Brownian motion with a power law drift.

Specialize $X_t$ to be a Brownian motion with a power law drift: the process satisfying the stochastic differential equation

$$dX_t = dB_t - \beta X_t^{-p}\, dt, \quad X_0 = x > 0,$$

where $\beta \neq 0$ and $p > 0$.

Note that when $p = 0$, $X_t$ is a Brownian motion with drift $-\beta$, for which is known that the hitting time of a certain level is a.s. finite if and only if the drift and the level have the same sign ([18]).

When $\beta = 0$, $X_t$ is merely Brownian motion, for which it is well-known (Feller [12]) that

$$P_x(\tau_0(X) > t) = \frac{2}{\sqrt{2\pi}} \int_0^{x/\sqrt{t}} e^{-u^2/2}\, dt.$$

When $p = 1$, $X_t$ is a Bessel process with dimension $\delta = 1 - 2\beta$, and it is known that for $\beta > -\frac{1}{2}$,

$$P_x(\tau_0(X) > t) = \frac{2^{1/2-\beta}}{\Gamma(1/2+\beta)} \int_0^{x/\sqrt{t}} u^{2\beta} e^{-u^2/2}\, dt,$$

and for $\beta \leq -\frac{1}{2}$, $P_x(\tau_0(X) = \infty) = 1$ (Göing-Jaeschke and Yor [14]).

Using standard results from diffusion theory (Theorem 1.1 of Chapter 5 in Pinsky [23]), it is not hard to show that if $\beta < 0$, then

$$\begin{cases} P_x(\tau_0(X) = \infty) = 1, & \text{for } p > 1 \\ 0 < P_x(\tau_0(X) = \infty) < 1, & \text{for } p < 1. \end{cases}$$

DeBlassie and Smits [10] proved the following results for $\beta > 0$.

- For $p > 1$,

$$\begin{cases} E_x[\tau_0(X)^q] < \infty, & \text{if } q < 1/2, \\ E_x[\tau_0(X)^q] = \infty, & \text{if } q > 1/2. \end{cases}$$

- For $p < 1$, $\lim_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) = -\gamma(p,\beta)$, where

$$\gamma(p,\beta) = \frac{1}{2} p^{-2p/(1+p)} \beta^{2/(1+p)} \left[ B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B\left(\frac{3}{2}, \frac{1-p}{2p}\right)\right] B\left(\frac{1}{2}, \frac{1-p}{2p}\right)^{-(1-p)/(1+p)},$$

  with $B$ denoting the Beta function, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$.

In particular, there is a phase transition of sorts as the power $p$ passes through the value 1: roughly power law behavior (with the same power) of the tail versus subexponential behavior (with a different power).

For the remaining case of $\beta > 0$ and $0 < p < 1$, it is not hard to show (we do so in the next section) that as long as the form of the drift is $\beta x^{-p}$ for large values of $x$, one can change the drift to be of the form $\alpha x^{-q}$, $0 < q < 1$, for small values of $x$, keeping it bounded in between, and still have the process hit 0 almost surely. This holds even if the multiplier $\alpha$ is negative. A natural question suggested by this is to determine how much effect this change has on the asymptotic behavior of the time to hit 0.

We now state our results precisely. Consider the diffusion $X_t$ given by

$$dX_t = dB_t + b(X_t)\,dt, \quad X_0 = x > 0, \tag{1.1}$$

where $B_t$ is one-dimensional Brownian motion and for some $0 < M_1 < M_2$, and $0 < p, q < 1$,

$$b(x) = \begin{cases} -\alpha x^{-q}, & 0 < x \le M_1 \\ \text{bounded measurable}, & M_1 < x < M_2, \quad \alpha \in \mathbb{R}, \beta > 0. \\ -\beta x^{-p}, & M_2 \le x \end{cases} \tag{1.2}$$

By standard facts, the law of $X_t$ on the space of continuous paths in $(0,\infty)$ exists uniquely up to an explosion time. Below, we will show the explosion time is the first hitting time at 0 and is a.s. finite:

$$P_x(\tau_0(X) < \infty) = 1. \tag{1.3}$$

Here is our main theorem.

**Theorem 1.1.** *Under the condition* (1.2)*, the solution $X_t$ of* (1.1) *satisfies*

$$\lim_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) = -\gamma(p,\beta),$$

*where*

$$\gamma(p,\beta) = \tfrac{1}{2} p^{-2p/(1+p)} \beta^{2/(1+p)} \left[ B\left(\tfrac{1}{2}, \tfrac{1-p}{2p}\right) + B\left(\tfrac{3}{2}, \tfrac{1-p}{2p}\right) \right] B\left(\tfrac{1}{2}, \tfrac{1-p}{2p}\right)^{-(1-p)/(1+p)}. \tag{1.4}$$

It is surprising that the rate function $\gamma$ is independent of $\alpha$ and $q$. This says the behavior of the process near the origin does not influence the time to hit 0 over very long time periods, at least at the logarithmic level. This is surprising because a negative $\alpha$ would push the process away from 0, hence delaying the time to exit, while a positive $\alpha$ would push the process towards zero, hence decreasing the time to exit.

Our result extends to drifts of the form $\alpha(x)x^q\ell_1(x)$ near 0 and $\beta(x)x^p\ell_2(x)$ near $\infty$, where $\ell_1$ and $\ell_2$ are slowly varying at 0 and $\infty$, respectively, and the coefficients $\alpha(x)$ and $\beta(x)$ have limiting values $\alpha_0$ and $\beta_0 > 0$. In the context of regular variation, one would say $x^q\ell_1(x)$ is regularly varying at 0 from the right, with index $q$, and $x^p\ell_2(x)$ is regularly varying at infinity, with index $p$.

To be precise, a measurable function $\ell > 0$ is slowly varying at infinity if for each $\lambda > 0$, $\ell(\lambda x)/\ell(x) \to 1$, as $x \to \infty$. Similarly, a measurable function $\ell > 0$ is slowly varying at zero from the right if for each $\lambda > 0$, $\ell(\lambda x)/\ell(x) \to 1$, as $x \to 0^+$.

**Theorem 1.2.** *Suppose $\alpha : (0,\infty) \to \mathbb{R}$ and $\beta : (0,\infty) \to (0,\infty)$ are continuous and $p, q \in (0,1)$. Assume:*

- *$\alpha(x)$ is bounded in a neighborhood of 0 and $\lim_{x\to\infty} \beta(x) = \beta_0 > 0$ exists;*
- *$\ell_1$ is continuous and slowly varying at zero from the right;*
- *$\ell_2$ is continuous and slowly varying at infinity.*

*Suppose, for some $0 < M_1 < M_2$,*

$$b(x) = \begin{cases} -\alpha(x)x^q\ell_1(x), & 0 < x \le M_1 \\ \text{bounded measurable}, & M_1 < x < M_2 \\ -\beta(x)x^p\ell_2(x), & M_2 \le x. \end{cases} \tag{1.5}$$

*Then the solution $X_t$ of (1.1) satisfies*

$$\lim_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) = -\gamma(p,\beta_0),$$

*where $\gamma(p,\beta_0)$ is from (1.4), and is independent of the function $\alpha$ and the power $q$.*

Here is a roadmap to our proof of Theorem 1.1. The basic idea is to "bootstrap" from carefully chosen special cases to the general case. In Section 2, we prove almost sure finiteness of the explosion time $\tau_0(X)$ and we set the stage for the case $b \in C^3$ by deriving a Feynman-Kac representation of $P_x(\tau_0(X) > t)$ solely in terms of Brownian motion. It requires the use of the $h-$transform. For the convenience of the reader, we provide a synopsis of the relevant facts about the $h-$transform in §1 of the supplement paper [9]. There, in §2, we also list some variational formulas from [10].

In Section 3, we first obtain the lim inf behavior for the special case $b \in C^3$. For that, we use the Feynman-Kac representation of $P_x(\tau_0(X) > t)$ in terms of Brownian motion $B_t$ from Section 2. By scaling the representation, we reduce consideration to the scaled process $\sqrt{t}B_u$, $u \in [0,1]$. The lim inf behavior is derived by analyzing small, intermediate and large values of the scaled process. The argument is technical and nontrivial. The lim inf behavior in the general case follows using a simple comparison argument with a $C^3$ function below $b$.

The lim sup behavior is much harder to obtain. Even in the $C^3$ case, analysis of the small and intermediate values of the scaled process in the Feynman-Kac representation of $P_x(\tau_0(X) > t)$ in terms of Brownian motion $B_t$ seems impossible. The way around this is to impose the simple extra condition $b' \geq 0$ to eliminate the issue. This reduces the analysis to only large values of the scaled process. Together with some scaling tricks, this allows us to convert the estimation to a standard large deviations setup. This is the content of Section 4.

For the lim sup behavior in the general case, it is natural to try to mimic what was done for the lim inf behavior and compare a general $b$ with a nondecreasing $C^3$ function satisfying (1.2) lying above $b$. In general, this is impossible because past a certain point, $b$ is negative and there can be places before that point where $b$ is positive. In that case, it will be impossible to find a nondecreasing $C^3$ function satisfying (1.2) (with the corresponding $M_1$ and $M_2$ not necessarily matching those for $b$) that lies above $b$. The remedy is to extend the Feynman-Kac representation of $P_x(\tau_0(X) > t)$ to drifts $b$ that are not necessarily $C^3$ or nondecreasing. As is standard in deriving Feynman-Kac representations, the Girsanov Theorem is used to change drifts appropriately. We do this for different $b$'s that match near 0 and $\infty$, but not in between, and that are not necessarily $C^3$. This is the content of Lemma 5.2 in Section 5. Lemma 5.4 is a variant of this for drifts that match at $\infty$, but not necessarily near 0.

In Section 6, we choose appropriate "comparison drifts" to render a useful form of the Feynman-Kac representation from Section 5. This will give the lim sup behavior for the case when $b$ from (1.2) is positive and constant on $[M_1, M_2]$. A simple comparison argument yields the desired lim sup behavior for the general case.

In the sequel, for a stochastic process $Z_t$ in $(0,\infty)$, write $\tau_{\varepsilon,M}(Z) = \tau_\varepsilon(Z) \wedge \tau_M(Z)$, $0 < \varepsilon < M$. $B_t$ will denote a generic Brownian motion and $B_0$ will be clear from context.

## 2. Preliminaries

First we prove zero is hit almost surely.

**Lemma 2.1.** *For $X_t$ as in (1.1) with $b$ as in (1.2), $P_x(\tau_0(X) < \infty) = 1$ for all $x > 0$.*

This will be a consequence of the following special case.

**Lemma 2.2.** *For $X_t$ from (1.1), where $b$ from (1.2) is continuous, $P_x(\tau_0(X) < \infty) = 1$ for all $x > 0$.*

**Proof.** Let $c \in (0, M_1)$ and for $x \in (0, \infty)$ set

$$p(x) = \int_c^x \exp\left(-2\int_c^\xi b(\zeta)d\zeta\right)d\xi \quad \text{(scale function), and}$$
$$v(x) = \int_c^x \exp\left(-2\int_c^y b(\zeta)d\zeta\right)\left[\int_c^y \exp\left(2\int_c^z b(\xi)d\xi\right)dz\right]dy.$$

Routine computations show $p(\infty) = \lim_{x\to\infty} p(x) = \infty$ (since $\beta > 0$), $p(0^+) = \lim_{x\to 0^+} p(x) < \infty$ and $v(0^+) < \infty$. Then by Feller's test for explosions, Proposition 5.32 (iii) in Karatzas and Shreve ([18]), the explosion time $S = \inf\{t \geq 0 : X_t \notin (0,\infty)\}$ of $X_t$ is a.s finite. By Proposition 5.22($b$) in ([18]), $\lim_{t\uparrow S} X_t = 0$ a.s. It follows that $S = \tau_0(X)$ a.s. and so $P_x(\tau_0(X) < \infty) = 1$. □

**Proof of Lemma 2.1.** Given $b$ as in (1.2), since $b$ is bounded on $[M_1, M_2]$, we can choose $\widetilde{M_1} < M_1$ and $\widetilde{M_2} > M_2$, along with a corresponding continuous $\tilde{b}$ satisfying (1.2) for $\widetilde{M_1}$, $\widetilde{M_2}$, such that $b \leq \tilde{b}$ on $(0,\infty)$. Then for $d\widetilde{X}_t = dB_t + \tilde{b}(\widetilde{X}_t)\,dt$, $\widetilde{X}_0 = x$, by the Comparison Theorem for SDEs (Ikeda and Watanabe [17], Theorem 1.1 in Chapter VI), $X_t \leq \widetilde{X}_t$ almost surely and so

$$1 = P_x(\tau_0(\widetilde{X}) < \infty) \leq P_x(\tau_0(X) < \infty),$$

and so $P_x(\tau_0(X) < \infty) = 1$, as claimed. □

The next results pertain to the case when $b \in C^3(0,\infty)$.

**Lemma 2.3.** *Suppose $b$ from (1.2) is in $C^1$. Then for*

$$h(x) = \exp\left(\int_0^x b(y)\,dy\right),\ x > 0, \tag{2.1}$$

$$h(x) \textit{ is bounded on } [M_1, M_2]; \tag{2.2}$$

$$h(x) \textit{ is bounded below away from } 0 \textit{ on } [M_1, M_2]; \tag{2.3}$$

$$h(x) = \exp\left(-\tfrac{\alpha}{1-q}x^{1-q}\right) \textit{ on } (0, M_1]; \tag{2.4}$$

$$h(x) = C\exp\left(-\tfrac{\beta}{1-p}x^{1-p}\right),\ x > M_2, \textit{ for some positive constant } C. \tag{2.5}$$

**Proof.** For some constants $C_1$ and $C_2$,

$$\int_0^x b(y)\,dy = -\tfrac{\alpha}{1-q}x^{1-q}I_{(0,M_1]}(x) + \left[C_1 + \int_{M_1}^x b(y)\,dy\right]I_{(M_1,M_2)}(x)$$
$$+\left[C_2 - \tfrac{\beta}{1-p}x^{1-p}\right]I_{[M_2,\infty)}(x). \tag{2.6}$$

The assertions of the Lemma follow immediately from this. □

**Lemma 2.4.** *Suppose* $b$ *from* (1.2) *is in* $C^1$ *and* $\alpha \geq 0$. *Then for*

$$V = -\tfrac{1}{2}(b^2 + b'), \tag{2.7}$$

*we have* $V \leq V I_{(M_1,\infty)}$.

**Proof.** Since $\alpha \geq 0$ and $0 < q < 1$, on the interval $(0, M_1]$ we have $V(x) = -\frac{\alpha^2}{2x^{2q}} - \frac{\alpha q}{2x^{q+1}} \leq 0$. The conclusion is immediate. □

The next result is a Feynman-Kac representation of $P_x(\tau_0(X) > t)$ solely in terms of Brownian motion.

**Lemma 2.5.** *Suppose* $X_t$ *is from* (1.1) *in the Introduction, where* $X_0 = x > 0$ *and* $b$ *given by* (1.2) *is in* $C^3$. *Then for* $h$ *from* (2.1) *and* $V$ *from* (2.7),

$$P_x(\tau_0(X) > t) = E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right) h(B_t) I_{\tau_0(B)>t}\right].$$

**Proof.** Let $L = \frac{1}{2}\frac{d^2}{dx^2} + V$. Note $h$ is positive and $L$-harmonic on $(0,\infty)$. The $h$-transform $L^h$ of $L$ is defined by $L^h f = \frac{1}{h} L(hf)$. It is easy to show that $L^h = \frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx}$. Next, we will use the following identity from §4 of Pinsky [23], which is stated as Theorem 1.1 in our Supplement [9], and we apply it here for a domain $D \subseteq \mathbb{R}^d$ and $f \in C_0(D)$, to get

$$\frac{1}{h(x)} E^{Q_x}\left[\exp\left(\int_0^t V(Y_s)\,ds\right)(hf)(Y_t) I_{\tau_D(Y)>t}\right] = E^{Q_x^h}\left[\exp\left(\int_0^t \frac{Lh}{h}(Y_s)\,ds\right) f(Y_t) I_{\tau_D(Y)>t}\right].$$

Let $0 < \varepsilon < M < \infty$ be such that $x \in (\varepsilon, M)$ and in the above identity take $d = 1$, $D = (\varepsilon, M)$. Then, under the measure $Q_x$ on $C([0,\infty), (\varepsilon, M))$, the coordinate process $Y_t$ satisfies $dY_t = dB_t$, $Y_0 = x$, and under the measure $Q_x^h$, the coordinate process satisfies $dY_t = dB_t + \frac{h'}{h}(Y_t)dt$, $Y_0 = x$. Thus, for our process $dX_t = dB_t + b(X_t)dt$, $X_0 = x$, under the underlying $P_x$, the above identity yields for $f \in C_0(\varepsilon, M)$,

$$\begin{aligned}
&\frac{1}{h(x)} E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right)(hf)(B_t) I_{\tau_{\varepsilon,M}(B)>t}\right] \\
&= \frac{1}{h(x)} E^{Q_x}\left[\exp\left(\int_0^t V(B_s)\,ds\right)(hf)(Y_t) I_{\tau_D(Y)>t}\right] \\
&= E^{Q_x^h}\left[\exp\left(\int_0^t \frac{Lh}{h}(Y_s)\,ds\right) f(Y_t) I_{\tau_D(Y)>t}\right] \\
&= E^{Q_x^h}\left[f(Y_t) I_{\tau_D(Y)>t}\right] \text{ (since } Lh = 0) = E_x\left[f(X_t) I_{\tau_{\varepsilon,M}(X)>t}\right].
\end{aligned}$$

By Monotone Convergence and Lemma 2.1, we can let $\varepsilon \downarrow 0$ and $M \uparrow \infty$ to get

$$\frac{1}{h(x)} E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right) h(B_t) I_{\tau_0(B)>t}\right] = P_x(\tau_0(X) > t),$$

as claimed. □

# 3. Lower bound

The main result of this section is the following lower bound.

**Theorem 3.1.** *Let $X_t$ be as in (1.1), where $b$ is from (1.2). Then*

$$\liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \geq -\gamma(p,\beta),$$

*where $\gamma(p,\beta)$ is from (1.4).*

We first prove a special case.

**Theorem 3.2.** *Let $X_t$ be as in (1.1), where $b$ from (1.2) satisfies $\alpha \geq 0$ and $b \in C^3$. Then*

$$\liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \geq -\gamma(p,\beta),$$

*where $\gamma(p,\beta)$ is from (1.4).*

The proof of the theorem is long, so we break it up into pieces. By Lemma 2.5

$$P_x(\tau_0(X) > t) = E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right) h(B_t) I_{\tau_0(B)>t}\right]. \tag{3.1}$$

Define

$$K_0 = \{\omega \in C_0 : \int_0^1 |\omega_u'|^2\,du < \infty\},$$

where $C_0 = \{\omega : [0,1] \to \mathbb{R},\ \omega \text{ is continuous}, \omega(0) = 0\}$, and

$$F(\omega) = \frac{\beta^2}{2}\int_0^1 |\omega_u|^{-2p}\,du + \frac{\beta}{1-p}|\omega_1|^{1-p} + \frac{1}{2}\int_0^1 (\omega_u')^2\,du, \quad \omega \in K_0. \tag{3.2}$$

Let $g \in K_0$ with $g \geq 0$. Let $\delta > 0$ be given and set $\tilde{g} = g + \delta$,

$$\varepsilon = t^{-(1-p)/(1+p)}, \tag{3.3}$$

and

$$Z_u = B_u - \tilde{g}(u)/\sqrt{\varepsilon}. \tag{3.4}$$

For $Z_0 > 0$, we have

$$\tau_0(Z) > 1 \Longrightarrow \tau_0(B) > 1 \tag{3.5}$$

$$\tau_0(Z) > 1 \Longrightarrow B_u \geq \tilde{g}(u)/\sqrt{\varepsilon},\ u \in [0,1]. \tag{3.6}$$

Now scale and use (3.5):

$$\begin{aligned} P_x(\tau_0(X) > t) &= E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right) h(B_t) I_{\tau_0(B)>t}\right] \\ &= E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u)\,du\right) h(\sqrt{t}B_1) I_{\tau_0(B)>1}\right] \end{aligned}$$

$$\geq E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u)du\right)h(\sqrt{t}B_1)I_{\tau_0(Z)>1}\right]. \tag{3.7}$$

**Lemma 3.3.** *For $\tau_0(Z)>1$ and $\varepsilon$ from (3.3),*

$$t\int_0^1 V(\sqrt{t}B_u)\left[I_{\sqrt{t}B_u<M_1}+I_{M_2<\sqrt{t}B_u}\right]du \geq G_1(\tilde g,\varepsilon),\ \text{where}\ \lim_{\varepsilon\to 0}\varepsilon\, G_1(\tilde g,\varepsilon)=-\frac{\beta^2}{2}\int_0^1 \tilde g(u)^{-2p}\,du.$$

**Proof.** By (2.7),

$$V(x)\left[I_{0<x<M_1}+I_{M_2<x}\right]=-\left(\frac{\alpha^2}{2x^{2q}}+\frac{q\alpha}{2x^{q+1}}\right)I_{0<x<M_1}-\left(\frac{\beta^2}{2x^{2p}}+\frac{p\beta}{2x^{p+1}}\right)I_{M_2<x}.$$

This is nonpositive and increasing, since $\alpha\geq 0$ and $\beta>0$. By (3.6), for $u\in[0,1]$, $B_u\geq\frac{\tilde g(u)}{\sqrt{\varepsilon}}$, and so $V\left(\sqrt{t}B_u\right)\geq V\left(\sqrt{\frac{t}{\varepsilon}}\,\tilde g(u)\right)$. Thus for $\tau_0(Z)>1$,

$$\begin{aligned}
t\int_0^1 V(\sqrt{t}B_u)\left[I_{\sqrt{t}B_u<M_1}+I_{M_2<\sqrt{t}B_u}\right]du &\geq t\int_0^1 V\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)\left[I_{\sqrt{t}B_u<M_1}+I_{M_2<\sqrt{t}B_u}\right]du\\
&=-t\int_0^1\left[\frac{\alpha^2}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-2q}+\frac{q\alpha}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-q-1}\right]I_{\sqrt{t}B_u<M_1}\,du\\
&\quad -t\int_0^1\left[\frac{\beta^2}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-2p}+\frac{p\beta}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-p-1}\right]I_{M_2<\sqrt{t}B_u}\,du\\
&\geq -t\int_0^1\left[\frac{\alpha^2}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-2q}+\frac{q\alpha}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-q-1}\right]I_{\sqrt{t/\varepsilon}\tilde g(u)<M_1}\,du\\
&\quad -t\int_0^1\left[\frac{\beta^2}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-2p}+\frac{p\beta}{2}\left(\sqrt{\tfrac{t}{\varepsilon}}\,\tilde g(u)\right)^{-p-1}\right]du
\end{aligned}$$

(since $I_{M_2<\sqrt{t}B_u}\leq 1$ and since $\tau_0(Z)>1$ implies $I_{\sqrt{t}B_u<M_1}=I_{\sqrt{t}(Z_u+\tilde g(u)/\sqrt{\varepsilon})<M_1}\leq I_{\sqrt{t/\varepsilon}\tilde g(u)<M_1}$ for $u\in[0,1]$)

$$\begin{aligned}
&=-\frac{1}{2}\int_0^1\left[\alpha^2\varepsilon^{(2q-p-1)/(1-p)}\tilde g(u)^{-2q}+q\alpha\varepsilon^{(q-p)/(1-p)}\tilde g(u)^{-q-1}\right]I_{\tilde g(u)<M_1\varepsilon^{1/(1-p)}}\,du\\
&\quad-\frac{1}{2}\int_0^1\left[\beta^2\varepsilon^{-1}\tilde g(u)^{-2p}+p\beta\tilde g(u)^{-p-1}\right]du,
\end{aligned} \tag{3.8}$$

where we have used (3.3) to replace the variable $t$ in terms of the variable $\varepsilon$.

Define $G_1(\tilde g,\varepsilon)=\mathrm{RHS}(3.8)$. Then

$$\begin{aligned}
\varepsilon\, G_1(\tilde g,\varepsilon)=&-\frac{1}{2}\int_0^1\left[\alpha^2\varepsilon^{2(q-p)/(1-p)}\tilde g(u)^{-2q}+q\alpha\varepsilon^{(1+q-2p)/(1-p)}\tilde g(u)^{-q-1}\right]I_{\tilde g(u)<M_1\varepsilon^{1/(1-p)}}\,du\\
&-\frac{1}{2}\int_0^1\left[\beta^2\tilde g(u)^{-2p}+p\beta\varepsilon\tilde g(u)^{-p-1}\right]du.
\end{aligned} \tag{3.9}$$

Now if $\varepsilon < (\delta/M_1)^{1-p}$, then for any $u \in [0,1]$, $\tilde{g}(u) \ge \delta > M_1 \varepsilon^{1/(1-p)}$. Thus the first integral in (3.9) is 0. The integrand in the second integral is bounded because $\tilde{g} \ge \delta$ and $p > 0$. It follows that

$$\lim_{\varepsilon \to 0} \varepsilon\, G_1(\tilde{g}, \varepsilon) = -\frac{1}{2} \int_0^1 \beta^2 \tilde{g}(u)^{-2p}\, du,$$

as claimed. □

**Lemma 3.4.** *For any positive $C_1$ and $\gamma$, as $\varepsilon \to 0$ (where $\varepsilon$ is from (3.3)),*

$$P_{x/\sqrt{t}}\left( C_1 t \int_0^1 I_{\sqrt{t} B_u + \sqrt{t/\varepsilon}\, \tilde{g}(u) < M_2}\, du + \frac{\beta}{1-p} \left(\sqrt{t} B_1\right)^{1-p} + \frac{1}{\sqrt{\varepsilon}} \int_0^1 \tilde{g}'(u)\, dB_u > \frac{\gamma}{\varepsilon},\ \tau_0(B) > 1 \right)$$
$$= o\left( P_{x/\sqrt{t}}\left(\tau_0(B) > 1\right)\right).$$

**Proof.** Since $\tau_0(B) > 1$ implies $B_u > 0$ for $u \in [0,1]$,

$$\begin{aligned}
&P_{x/\sqrt{t}}\left( C_1 t \int_0^1 I_{\sqrt{t} B_u + \sqrt{t/\varepsilon}\, \tilde{g}(u) < M_2}\, du > \frac{\gamma}{3\varepsilon},\ \tau_0(B) > 1 \right) \\
&\quad \le e^{-\gamma/3\varepsilon} E_{x/\sqrt{t}}\left[ \exp\left( C_1 t \int_0^1 I_{\sqrt{t} B_u + \sqrt{t/\varepsilon}\, \tilde{g}(u) < M_2}\, du \right) I_{\tau_0(B) > 1} \right] \\
&\quad \le e^{-\gamma/3\varepsilon} E_{x/\sqrt{t}}\left[ \exp\left( C_1 t \int_0^1 I_{\sqrt{t/\varepsilon}\, \tilde{g}(u) < M_2}\, du \right) I_{\tau_0(B) > 1} \right] \\
&\quad \le e^{-\gamma/3\varepsilon} \exp\left( C_1 t \int_0^1 I_{\sqrt{t/\varepsilon}\, \delta < M_2}\, du \right) P_{x/\sqrt{t}}(\tau_0(B) > 1), \qquad (3.10)
\end{aligned}$$

(since $\tilde{g} \ge \delta$). But by (3.3), $\dfrac{t}{\varepsilon} = \varepsilon^{-(1+p)/(1-p)-1} \to \infty$ as $\varepsilon \to 0$, so for small $\varepsilon$, $I_{\sqrt{t/\varepsilon}\, \delta < M_2} = 0$ and we get

$$P_{x/\sqrt{t}}\left( C_1 t \int_0^1 I_{\sqrt{t} B_u + \sqrt{t/\varepsilon}\, \tilde{g}(u) < M_2}\, du > \frac{\gamma}{3\varepsilon},\ \tau_0(B) > 1 \right) \le e^{-\gamma/3\varepsilon}. \qquad (3.11)$$

Next, for some constant $C$ (whose exact value might change from line to line), independent of $\varepsilon$,

$$\begin{aligned}
&P_{x/\sqrt{t}}\left( \frac{\beta}{1-p} \left(\sqrt{t} B_1\right)^{1-p} > \frac{\gamma}{3\varepsilon} \right) = P_{x/\sqrt{t}}\left( B_1 > C \left(\frac{\gamma}{\varepsilon}\right)^{1/(1-p)} t^{-1/2} \right) \\
&\quad \le \exp\left( -C \varepsilon^{-1/(1-p)} t^{-1/2} \right) E_{x/\sqrt{t}}\left[ e^{B_1} \right] = \exp\left( -C\varepsilon^{-1/2} + x \varepsilon^{(1+p)/2(1-p)} + 1/2 \right), \qquad (3.12)
\end{aligned}$$

using (3.3) to write $t$ in terms of $\varepsilon$. Finally,

$$\begin{aligned}
&P_{x/\sqrt{t}}\left( \frac{1}{\sqrt{\varepsilon}} \int_0^1 \tilde{g}'(u)\, dB_u > \frac{\gamma}{3\varepsilon} \right) \le e^{-\gamma/3\sqrt{\varepsilon}} E_{x/\sqrt{t}}\left[ \exp\left( \int_0^1 \tilde{g}'(u)\, dB_u \right) \right] \\
&\quad = e^{-\gamma/3\sqrt{\varepsilon}} \exp\left( \frac{1}{2} \int_0^1 \tilde{g}'(u)^2\, du \right), \qquad (3.13)
\end{aligned}$$

since under $P_{x/\sqrt{t}}$, $\int_0^1 \tilde{g}'(u)\, dB_u$ is normal with mean 0 and variance $\int_0^1 \tilde{g}'(u)^2\, du$.

Combining (3.11)–(3.13), we get

$$\begin{aligned}
&P_{x/\sqrt{t}}\left(C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\,\tilde{g}(u)<M_2}\,du+\frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}+\frac{1}{\sqrt{\varepsilon}}\int_0^1\tilde{g}'(u)\,dB_u>\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)\\
&\quad\le P_{x/\sqrt{t}}\left(C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\,\tilde{g}(u)<M_2}\,du>\frac{\gamma}{3\varepsilon},\,\tau_0(B)>1\right)\\
&\quad+P_{x/\sqrt{t}}\left(\frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}>\frac{\gamma}{3\varepsilon},\,\tau_0(B)>1\right)+P_{x/\sqrt{t}}\left(\frac{1}{\sqrt{\varepsilon}}\int_0^1\tilde{g}'(u)\,dB_u>\frac{\gamma}{3\varepsilon},\,\tau_0(B)>1\right)\\
&\quad\le e^{-\gamma/3\varepsilon}+C_2\exp\left(-C\varepsilon^{-1/2}+x\varepsilon^{(1+p)/2(1-p)}\right)+C_3e^{-\gamma/3\sqrt{\varepsilon}},
\end{aligned}\tag{3.14}$$

where the constants are independent of $\varepsilon$. From Feller [12], as $t\to\infty$ (or equivalently, as $\varepsilon\to 0$)

$$P_{x/\sqrt{t}}(\tau_0(B)>1)=\frac{2}{\sqrt{2\pi}}\int_0^{x/\sqrt{t}}e^{-u^2/2}\,du\sim\frac{2}{\sqrt{2\pi}}\frac{x}{\sqrt{t}}=C_4\,\varepsilon^{(1+p)/2(1-p)},\tag{3.15}$$

where $C_4$ is independent of $\varepsilon$ and we have used (3.3) to write $t$ in terms of $\varepsilon$.

Combining (3.14)–(3.15) yields the conclusion of the Lemma. □

**Lemma 3.5.** *For any $C_1>0$ and $\gamma>0$, with $\varepsilon>0$ from (3.3),*

$$\lim_{\varepsilon\to 0}\varepsilon\log P_{x/\sqrt{t}}\left(C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}du+\frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}+\frac{1}{\sqrt{\varepsilon}}\int_0^1\tilde{g}'(u)dB_u\le\frac{\gamma}{\varepsilon},\tau_0(B)>1\right)=0.$$

**Proof.** Write

$$M=C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\,du+\frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}+\frac{1}{\sqrt{\varepsilon}}\int_0^1\tilde{g}'(u)\,dB_u.$$

Then we want to show

$$\lim_{\varepsilon\to 0}\varepsilon\log P_{x/\sqrt{t}}\left(M\le\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)=0.\tag{3.16}$$

Notice

$$\begin{aligned}
P_{x/\sqrt{t}}\left(M\le\frac{\gamma}{\varepsilon},\ \tau_0(B)>1\right)&=P_{x/\sqrt{t}}(\tau_0(B)>1)-P_{x/\sqrt{t}}\left(M>\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)\\
&=P_{x/\sqrt{t}}(\tau_0(B)>1)\left[1-P_{x/\sqrt{t}}\left(M>\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)\Big/P_{x/\sqrt{t}}(\tau_0(B)>1)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\varepsilon\log P_{x/\sqrt{t}}\left(M\le\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)&=\varepsilon\log P_{x/\sqrt{t}}(\tau_0(B)>1)\\
&+\varepsilon\log\left[1-P_{x/\sqrt{t}}\left(M>\frac{\gamma}{\varepsilon},\,\tau_0(B)>1\right)\Big/P_{x/\sqrt{t}}(\tau_0(B)>1)\right].
\end{aligned}$$

Equation (3.16) follows from this, using Lemma 3.4 and (3.15). □

**Lemma 3.6.** *For any $C_1 > 0$, with $\varepsilon > 0$ from (3.3),*

$$\liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(-C_1 t \int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\, du - \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p} - \frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\, dB_u\right) I_{\tau_0(B)>1}\right] \geq 0.$$

**Proof.** For any $\gamma > 0$,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(-C_1 t \int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\, du - \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p} - \frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\, dB_u\right) I_{\tau_0(B)>1}\right] \\
&\geq \liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[e^{-\gamma/\varepsilon} I\left(C_1 t \int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\, du + \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p} + \frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\, dB_u \leq \frac{\gamma}{\varepsilon}\right) I_{\tau_0(B)>1}\right] \\
&= \liminf_{\varepsilon\to 0}\left[-\gamma + \varepsilon \log P_{x/\sqrt{t}}\left(C_1 t \int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\, du + \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p} + \frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\, dB_u \leq \frac{\gamma}{\varepsilon}, \tau_0(B) > 1\right)\right] \geq -\gamma + 0,
\end{aligned}$$

by Lemma 3.5. Let $\gamma \to 0$ to finish. □

**Lemma 3.7.** *We have for $\varepsilon > 0$ from (3.3),*

$$\liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\, du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right] \geq -\frac{\beta}{1-p}\tilde{g}(1)^{1-p} - \frac{1}{2}\int_0^1 \tilde{g}'(u)^2 du.$$

**Proof.** Since $V$ is bounded on $[M_1, M_2]$, there is $C_1 > 0$ such that $|V| \leq C_1$ on that interval. Then

$$\begin{aligned}
&E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\, du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right] \\
&\geq E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\, du\right) h(\sqrt{t}B_1) I_{M_2<\sqrt{t}B_1} I_{\tau_0(Z)>1}\right] \\
&\geq C E_{x/\sqrt{t}}\left[\exp\left(-C_1 t\int_0^1 I_{\sqrt{t}B_u<M_2}\, du - \frac{\beta}{1-p}(\sqrt{t}B_1)^{1-p}\right) I_{M_2<\sqrt{t}B_1} I_{\tau_0(Z)>1}\right]
\end{aligned}$$

(where $C$ is from (2.5) and using that $I_{M_1<\sqrt{t}B_u<M_2} \leq I_{\sqrt{t}B_u<M_2}$)

$$\geq C E_{x/\sqrt{t}}\left[\exp\left(-C_1 t\int_0^1 I_{\sqrt{t}Z_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\, du - \frac{\beta}{1-p}\left(\sqrt{t}Z_1 + \sqrt{t/\varepsilon}\tilde{g}(1)\right)^{1-p}\right).$$

$$\cdot I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)} I_{\tau_0(Z)>1}\Big]$$

(using (3.4) and that $I_{M_2<\sqrt{t}B_1} = I_{M_2<\sqrt{t}Z_1+\sqrt{t/\varepsilon}\tilde{g}(1)} \geq I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)}$ for $\tau_0(Z) > 1$)

$$\geq C E_{x/\sqrt{t}}\left[\exp\left(-C_1 t\int_0^1 I_{\sqrt{t}Z_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\,du - \frac{\beta}{1-p}\left(\sqrt{t}Z_1\right)^{1-p}\right.\right.$$
$$\left.\left.-\frac{\beta}{1-p}\left(\sqrt{t/\varepsilon}\,\tilde{g}(1)\right)^{1-p}\right) I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)} I_{\tau_0(Z)>1}\right]$$

(using that $(a+b)^{1-p} \leq a^{1-p} + b^{1-p}$ since $0 < p < 1$)

$$= C\exp\left(-\frac{\beta}{1-p}\left(\sqrt{t/\varepsilon}\,\tilde{g}(1)\right)^{1-p}\right) I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)}\cdot$$
$$\cdot E_{x/\sqrt{t}}\left[\exp\left(-C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\,du - \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}\right.\right.$$
$$\left.\left.-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\,dB_u - \frac{1}{2\varepsilon}\int_0^1 \tilde{g}'(u)^2\,du\right) I_{\tau_0(B)>1}\right]$$

(by the Cameron-Martin-Girsanov Theorem)

$$= C\exp\left(-\frac{\beta}{1-p}\left(\sqrt{t/\varepsilon}\,\tilde{g}(1)\right)^{1-p} - \frac{1}{2\varepsilon}\int_0^1 \tilde{g}'(u)^2\,du\right) I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)}\cdot$$
$$\cdot E_{x/\sqrt{t}}\left[\exp\left(-C_1 t\int_0^1 I_{\sqrt{t}B_u+\sqrt{t/\varepsilon}\tilde{g}(u)<M_2}\,du - \frac{\beta}{1-p}\left(\sqrt{t}B_1\right)^{1-p}\right.\right.$$
$$\left.\left.-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \tilde{g}'(u)\,dB_u\right) I_{\tau_0(B)>1}\right]. \tag{3.17}$$

From this and Lemma 3.6, we get

$$\liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\,du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right]$$
$$\geq \liminf_{\varepsilon\to 0} \varepsilon\left(\log C - \frac{\beta}{1-p}\left(\sqrt{t/\varepsilon}\,\tilde{g}(1)\right)^{1-p} - \frac{1}{2\varepsilon}\int_0^1 \tilde{g}'(u)^2\,du + \log I_{M_2<\sqrt{t/\varepsilon}\tilde{g}(1)}\right) + 0$$
$$= \liminf_{\varepsilon\to 0}\left(\varepsilon\log C - \frac{\beta}{1-p}\tilde{g}(1)^{1-p} - \frac{1}{2}\int_0^1 \tilde{g}'(u)^2\,du + \varepsilon\log I_{M_2<\varepsilon^{-1/(1+p)}\tilde{g}(1)}\right)$$

(using (3.3) to substitute for $t$ in terms of $\varepsilon$)

$$= -\frac{\beta}{1-p}\,\tilde{g}(1)^{1-p} - \frac{1}{2}\int_0^1 \tilde{g}'(u)^2\,du,$$

since $\lim_{\varepsilon\to 0} I_{M_2<\varepsilon^{-1/(1+p)}\tilde{g}(1)} = 1$, as $\tilde{g}(1) = g(1) + \delta > 0$. □

**Proof of Theorem 3.2.** By Theorem 2.1 in §2 of Supplement [9], $\gamma(p,\beta) = \inf_{\omega\in K_0,\,\omega\geq 0} F(\omega)$, where $F$ is from (3.2), so it suffices to show

$$\liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \geq -\inf_{\substack{g\in K_0\\ g\geq 0}} F(g). \tag{3.18}$$

To see why, let $g \in K_0$ with $g \geq 0$. By (3.3) and (3.7), (recalling $g \geq 0$ and $g \in C_0$),

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t)\\
&\quad\geq \liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(t\int_0^1 V(\sqrt{t}B_u)\,du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right], \text{ and by using Lemma 3.3},\\
&\quad\geq \liminf_{\varepsilon\to 0} \varepsilon \log E_{x/\sqrt{t}}\left[\exp\left(G_1(\tilde{g},\varepsilon) + t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\,du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right],\\
&\quad= \liminf_{\varepsilon\to 0}\left[\varepsilon G_1(\tilde{g},\varepsilon) + \varepsilon \log E_{x/\sqrt{t}}[\exp\left(t\int_0^1 V(\sqrt{t}B_u) I_{M_1<\sqrt{t}B_u<M_2}\,du\right) h(\sqrt{t}B_1) I_{\tau_0(Z)>1}\right]\\
&\quad\geq -\frac{\beta^2}{2}\int_0^1 \tilde{g}(u)^{-2p}\,du - \frac{\beta}{1-p}\tilde{g}(1)^{1-p} - \frac{1}{2}\int_0^1 \tilde{g}'(u)^2\,du, \text{ (by Lemmas 3.2 and 3.7)}.
\end{aligned}$$

Recalling $\tilde{g} = g + \delta$, where $\delta > 0$ was arbitrary, we can let $\delta \to 0$ to end up with

$$\liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \geq -\frac{\beta^2}{2}\int_0^1 g(u)^{-2p}du - \frac{\beta}{1-p} g(1)^{1-p} - \frac{1}{2}\int_0^1 g'(u)^2 du = -F(g).$$

Since $g \geq 0$ in $K_0$ was arbitrary, (3.18) holds. □

**Proof of Theorem 3.1.** Since $b$ is bounded below on $[M_1, M_2]$, we can choose $C < 0$ such that $C \leq b$ on $[M_1, M_2]$. Then for some $x_1 < M_1$ and $x_2 > M_2$, we can choose $\tilde{b} \in C^3(0,\infty)$ such that

$$\tilde{b}(x) = \begin{cases} -|\alpha| x^{-q}, & x \in (0, x_1] \\ -\beta x^{-p}, & x \in [x_2, \infty), \end{cases}$$

and $b \geq \tilde{b}$ on $(0,\infty)$. By the Comparison Theorem, if $dY_t = dB_t + \tilde{b}(Y_t)$, $Y_0 = x$, then

$$P_x(\tau_0(X) > t) \geq P_x(\tau_0(Y) > t).$$

The desired lower bound follows from Theorem 3.2 applied to $Y$. □

# 4. Upper bound: a special $C^3$ case

The main theorem of this section is the following special case. It will be crucial in proving the general case.

**Theorem 4.1.** *Let $X_t$ be from (1.1) where $b$ from (1.2) satisfies $\alpha \geq 0$, $b \in C^3$ and $b' \geq 0$ on $(0,\infty)$. Then, with $\gamma(p,\beta)$ from (1.4), we have*

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \leq -\gamma(p,\beta).$$

**Proof.** Let $h$ be from (2.1) and $V$ from (2.7). By Lemma 2.4 and our hypotheses that $b' \geq 0$ and $\alpha \geq 0$,

$$V(x) \leq V(x)I_{(M_1,\infty)}(x) = V(x)I_{[M_2,\infty)}(x) - \tfrac{1}{2}\left(b^2(x) + b'(x)\right) I_{(M_1,M_2)}(x) \leq V(x)I_{[M_2,\infty)}(x). \qquad (4.1)$$

By Lemma 2.3, since $\beta > 0$,

$$h(x) \leq C\exp\left(-\tfrac{\beta}{1-p}x^{1-p}\right), \qquad x > 0. \qquad (4.2)$$

By Lemma 2.5, (4.1), scaling and translation,

$$P_x(\tau_0(X)>t) = E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right)h(B_t)I_{\tau_0(B)>t}\right] \leq E_x\left[\exp\left(\int_0^t (VI_{[M_2,\infty)}(B_s))\,ds\right)h(B_t)I_{\tau_0(B)>t}\right]$$

$$= E_0\left[\exp\left(t\int_0^1 (VI_{[M_2,\infty)}(\sqrt{t}(B_u + x/\sqrt{t}))\,du\right)\cdot h(\sqrt{t}(B_1 + x/\sqrt{t}))I_{\tau_0(B+x/\sqrt{t})>1}\right]. \qquad (4.3)$$

Writing $\varepsilon = t^{-(1-p)/(1+p)}$ and $x_\varepsilon = x\varepsilon^{1/(1-p)}$, we have

$$\sqrt{t}(B_u + x/\sqrt{t}) = \sqrt{t}B_u + x = \varepsilon^{-1/(1-p)}(\varepsilon^{1/2}B_u + x_\varepsilon).$$

By (2.7) and (1.2),

$$VI_{[M_2,\infty)}(ax) = -\tfrac{1}{2}\left(\beta^2 a^{-2p}x^{-2p} + p\beta a^{-p-1}x^{-p-1}\right)I_{[a^{-1}M_2,\infty)}(x) \leq -\tfrac{1}{2}\beta^2 a^{-2p}x^{-2p}\,I_{[a^{-1}M_2,\infty)}(x).$$

Thus

$$\begin{aligned}(VI_{[M_2,\infty)})(\sqrt{t}B_u + x) &= (VI_{[M_2,\infty)})(\varepsilon^{-1/(1-p)}(\varepsilon^{1/2}B_u + x_\varepsilon)) \\ &\leq -\tfrac{1}{2}\beta^2\varepsilon^{2p/(1-p)}(\varepsilon^{1/2}B_u + x_\varepsilon)^{-2p}\,I_{[\varepsilon^{1/(1-p)}M_2,\infty)}(\varepsilon^{1/2}B_u + x_\varepsilon). \qquad (4.4)\end{aligned}$$

By (4.2),

$$h(\sqrt{t}B_1 + x) \leq C\exp\left(-\tfrac{\beta}{1-p}(\sqrt{t}B_1 + x)^{1-p}\right) = C\exp\left(-\tfrac{\beta}{1-p}\varepsilon^{-1}(\varepsilon^{1/2}B_1 + x_\varepsilon)^{1-p}\right). \qquad (4.5)$$

Substituting $t = \varepsilon^{-(1+p)/(1-p)}$ into (4.3) and using (4.4)–(3.5) gives

$$\begin{aligned}P_x(\tau_0(X) > t) &\leq CE_0\Bigg[\exp\Bigg(-\tfrac{\beta^2}{2}\varepsilon^{-1}\int_0^1\left(\varepsilon^{1/2}B_u + x_\varepsilon\right)^{-2p} I_{[\varepsilon^{1/(1-p)}M_2,\infty)}(\varepsilon^{1/2}B_u + x_\varepsilon)\,du \\ &\qquad -\tfrac{\beta}{1-p}\varepsilon^{-1}\left(\varepsilon^{1/2}B_1 + x_\varepsilon\right)^{1-p}\Bigg)I_{\tau_0(\varepsilon^{1/2}B+x_\varepsilon)>1}\Bigg] \\ &\leq CE_0\Bigg[\exp\Bigg(-\tfrac{\beta^2}{2}\varepsilon^{-1}\int_0^1\left(\varepsilon^{1/2}B_u + x_\varepsilon\right)^{-2p} I_{[\varepsilon^{1/(1-p)}M_2,\infty)}(\varepsilon^{1/2}B_u + x_\varepsilon)\,du \\ &\qquad -\tfrac{\beta}{1-p}\varepsilon^{-1}\left|\varepsilon^{1/2}B_1 + x_\varepsilon\right|^{1-p}\Bigg)\Bigg].\end{aligned}$$

Writing $Q_\varepsilon$ for the law on $C([0,\infty),\mathbb{R})$ of $\sqrt{\varepsilon}B$ under $P_0$, this becomes

$$P_x(\tau_0(X)>t) \le CE^{Q_\varepsilon}\left[\exp\left(-\tfrac{\beta^2}{2}\varepsilon^{-1}\int_0^1 (\omega_u+x_\varepsilon)^{-2p}\, I_{[\varepsilon^{1/(1-p)}M_2,\infty)}(\omega_u+x_\varepsilon)\,du - \tfrac{\beta}{1-p}\varepsilon^{-1}\,|\omega_1+x_\varepsilon|^{1-p}\right)\right] = CE^{Q_\varepsilon}\left[\exp\left(-\tfrac{1}{\varepsilon}J_\varepsilon(\omega)\right)\right], \text{ where}$$

$$J_\varepsilon(\omega) = \frac{\beta^2}{2}\int_0^1 (\omega_u+x_\varepsilon)^{-2p}\, I_{[\varepsilon^{1/(1-p)}M_2,\infty)}(\omega_u+x_\varepsilon)\,du + \frac{\beta}{1-p}\,|\omega_1+x_\varepsilon|^{1-p}.$$

Set

$$J(\omega) = \frac{\beta^2}{2}\int_0^1 \omega_u^{-2p} I_{\omega_u\ge 0}\,du + \frac{\beta}{1-p}\omega_1^{1-p} I_{\omega_1\ge 0},$$

if the integral is finite, otherwise set $J(\omega)=\infty$. Then $J$ is lower semicontinuous on $C_0$, and if $\omega_n\to\omega$ in $C_0$ as $n\to\infty$, then $\liminf_{n\to\infty,\varepsilon\to 0^+} J_\varepsilon(\omega_n)\ge J(\omega)$. If $\omega\in K_0=\{\omega\in C_0 : \int_0^1|\omega_u'|^2\,du<\infty\}$, by Varadhan's Theorem ([27], Theorem 2.3),

$$\limsup_{\varepsilon\to 0^+}\varepsilon\log E^{Q_\varepsilon}\left[\exp\left(-\tfrac{1}{\varepsilon}J_\varepsilon(\omega)\right)\right] \le -\inf_{\omega\in C_0}\left[J(\omega)+\tfrac12\int_0^1(\omega_u')^2\,du\right] = -\inf_{\omega\in K_0}\left[J(\omega)+\tfrac12\int_0^1|\omega_u'|^2\,du\right].$$

By Theorem 2.1 in Supplement [9], the infimum is $\gamma(p,\beta)$. This completes the proof. □

## 5. Transformation of drift

In this section we set the stage to extend Theorem 4.1 to a drift with nonnegative $\alpha$ and $b$ that can take on positive values on $[M_1,M_2]$. There will be no increasing or $C^3$ conditions imposed. We will also set things up for the case of negative $\alpha$.

First we prove a variant of the formula in Lemma 2.5 that is applicable to discontinuous drifts.

**Lemma 5.1.** *Suppose $b_X$ and $b_Y$ satisfy (1.2) with the same $M_1$ and $M_2$. If $b_X$ and $b_Y$ are continuous on $(0,\infty)\backslash\{M_2\}$ and each restricted to $(M_1,M_2)$ has a $C^1$ extension to $[M_1,M_2]$, then the function*

$$g(y) = \int_0^y (b_X-b_Y)(z)\,dz \tag{5.1}$$

*is a linear combination of convex functions with generalized second derivative $\mu$ given by*

$$\mu(A) = (b_X-b_Y)(M_2-)\,\delta_{M_2}(A) - \int_A \left((b_X'-b_Y')\,I_{(M_1,M_2)}\right)(a)\,da \tag{5.2}$$

*for any Borel set $A\subseteq(0,\infty)$. Here $\delta_{M_2}$ is the unit point mass at $M_2$.*

**Proof.** By our assumptions on $b_X$ and $b_Y$, $b_X-b_Y$ is of bounded variation on any bounded open subset $U$ of $(0,\infty)$, hence $g$ is a linear combination of convex functions on $U$ (Roberts and Varberg [26], page 23). We now show its generalized second derivative is given by (5.2).

Since $g' = b_X - b_Y$ a.e., it suffices to show for each $H \in C_0^\infty(0,\infty)$,

$$\int (H'(b_X - b_Y))(a)\,da = \int H(a)\,d\mu(a). \tag{5.3}$$

To this end, let $H$ be so given. Then denoting the $C^1$ extensions of $b_X$, $b_Y$ on $(M_1, M_2)$ to $[M_1, M_2]$ by $\tilde{b}_X$, $\tilde{b}_Y$, respectively,

$$\int H'(b_X - b_Y) = \int H'(\tilde{b}_X - \tilde{b}_Y) I_{(M_1,M_2)}$$

(since $b_X = b_Y$ on $(0, M_1] \cup [M_2, \infty)$)

$$= \lim_{\varepsilon \to 0^+} \int H'(\tilde{b}_X - \tilde{b}_Y) I_{(M_1+\varepsilon, M_2-\varepsilon)}$$

$$= \lim_{\varepsilon \to 0^+} \left[ (H(\tilde{b}_X - \tilde{b}_Y))(M_2 - \varepsilon) - (H(\tilde{b}_X - \tilde{b}_Y))(M_1 + \varepsilon) - \int_{M_1+\varepsilon}^{M_2-\varepsilon} H(\tilde{b}'_X - \tilde{b}'_Y) \right]$$

$$= \lim_{\varepsilon \to 0^+} \left[ (H(b_X - b_Y))(M_2 - \varepsilon) - (H(b_X - b_Y))(M_1 + \varepsilon) - \int_{M_1+\varepsilon}^{M_2-\varepsilon} H(b'_X - b'_Y) \right]$$

$$= (H(b_X - b_Y))(M_2-) - \int_{M_1}^{M_2} H(b'_X - b'_Y)$$

(since $b_X, b_Y$ are continuous at $M_1$ and coincide there, and $\tilde{b}'_X - \tilde{b}'_Y = b'_X - b'_Y$ is bounded on $(M_1, M_2)$)

$$= H(M_2)(b_X - b_Y)(M_2-) - \int_{M_1}^{M_2} H(b'_X - b'_Y),$$

(since $H$ is continuous). This gives (5.2). □

**Lemma 5.2.** *Let $x > 0$ and let*

$$\begin{cases} dX_t = dB_t + b_X(X_t)\,dt, & X_0 = x \\ dY_t = dB_t + b_Y(Y_t)\,dt, & Y_0 = x \end{cases} \tag{5.4}$$

*be such that $b_X$ and $b_Y$ satisfy (1.2) with the same $M_1$ and $M_2$. If $b_X$ and $b_Y$ are continuous on $(0,\infty)\backslash\{M_2\}$ and each restricted to $(M_1, M_2)$ has a $C^1$ extension to $[M_1, M_2]$, then for $g$ as in (5.1) and $H = \left(b_X^2 - b_Y^2\right) - (b'_X - b'_Y)\, I_{(M_1,M_2)}$, we have*

$$P_x(\tau_0(X) > t) = E_x \left[ \exp\left( g(Y_t) - g(x) - \tfrac{1}{2} \int_0^t H(Y_s)\,ds - \tfrac{1}{2}(b_X - b_Y)(M_2-)\,\ell_t^{M_2}(Y) \right) I_{\tau_0(Y) > t} \right], \tag{5.5}$$

*where $\ell_t^{M_2}(Y)$ is the local time of $Y$ at $M_2$.*

**Proof.** The idea is similar to the one used by Pinsky [23] to get the formula cited in Lemma 2.5: use the Girsanov Theorem and eliminate the stochastic integral in the martingale measure. Our reference for the particular form of the Girsanov Theorem that we use is Theorem 8.6.6 in Oksendal [22].

Let $0 < \varepsilon < M_1 \le M_2 < M$ such that $x \in (\varepsilon, M)$. Then since $b_X - b_Y$ is bounded on $(\varepsilon, M)$, the "Local Novikov Condition" holds:

$$E_x\left[\exp\left(\tfrac{1}{2}\int_0^{t\wedge\tau_{\varepsilon,M}(Y)}(b_X-b_Y)^2(Y_s)\,ds\right)\right] < \infty.$$

It follows that

$$M_t = \exp\left(\int_0^{t\wedge\tau_{\varepsilon,M}(Y)}(b_X-b_Y)(Y_s)\,dB_s - \tfrac{1}{2}\int_0^{t\wedge\tau_{\varepsilon,M}(Y)}(b_X-b_Y)^2(Y_s)\,ds\right)$$

is a martingale. By Girsanov's Theorem,

$$\begin{aligned} P_x(\tau_{\varepsilon,M}(X) > t) &= E_x\left[M_t I_{\tau_{\varepsilon,M}(Y)>t}\right] \\ &= E_x\left[\exp\left(\int_0^t (b_X-b_Y)(Y_s)\,dB_s - \tfrac{1}{2}\int_0^t (b_X-b_Y)^2(Y_s)\,ds\right) I_{\tau_{\varepsilon,M}(Y)>t}\right]. \end{aligned} \tag{5.6}$$

Now we eliminate the stochastic integral in (5.6). By Lemma 5.1, we can apply the Itô-Tanaka formula (Revuz and Yor [25]) to get for $\tau_{\varepsilon,M}(Y) > t$,

$$g(Y_t) = g(x) + \int_0^t g'_-(Y_s)\,[dB_s + b_Y(Y_s)\,ds] + \tfrac{1}{2}\int \ell_t^a(Y)\mu(da),$$

where $g'_-$ is the left derivative and $\mu(da)$ is the generalized second derivative of $g$ from (5.2) in Lemma 5.1. By our hypotheses on $b_X$ and $b_Y$, $g'_- = b_X - b_Y$ a.e. on $(0,\infty)$. Using this, the occupation times formula (Revuz and Yor [25]) and Lemma 5.1 gives, for $\tau_{\varepsilon,M}(Y) > t$,

$$\begin{aligned} g(Y_t) = g(x) &+ \int_0^t (b_X-b_Y)(Y_s)\,[dB_s + b_Y(Y_s)\,ds] \\ &+ \tfrac{1}{2}\left[(b_X-b_Y)(M_2-)\,\ell_t^{M_2}(Y) - \int_0^t ((b'_X-b'_Y)I_{(M_1,M_2)})(Y_s)\,ds\right]. \end{aligned}$$

Solving for the stochastic integral, we get, for $\tau_{\varepsilon,M}(Y) > t$,

$$\begin{aligned} \int_0^t (b_X-b_Y)(Y_s)\,dB_s = g(Y_t) - g(x) &- \int_0^t ((b_X-b_Y)b_Y)(Y_s)\,ds \\ &+ \tfrac{1}{2}\int_0^t \left((b'_X-b'_Y)I_{(M_1,M_2)}\right)(Y_s)\,ds - \tfrac{1}{2}(b_X-b_Y)(M_2-)\,\ell_t^{M_2}(Y). \end{aligned}$$

Substituting this into the exponential in (5.6) gives

$$\begin{aligned} P_x(\tau_{\varepsilon,M(X)} > t) = E_x\Bigg[\exp\Bigg(g(Y_t) - g(x) &- \int_0^t ((b_X-b_Y)b_Y)(Y_s)\,ds \\ &+ \tfrac{1}{2}\int_0^t \left((b'_X-b'_Y)I_{(M_1,M_2)}\right)(Y_s)\,ds - \tfrac{1}{2}(b_X-b_Y)(M_2-)\,\ell_t^{M_2}(Y) \\ &- \tfrac{1}{2}\int_0^t (b_X-b_Y)^2(Y_s)\,ds\Bigg) I_{\tau_{\varepsilon,M}(Y)>t}\Bigg] \end{aligned}$$

$$= E_x\left[\exp\left(g(Y_t) - g(x) - \tfrac{1}{2}\int_0^t H(Y_s)\,ds - \tfrac{1}{2}(b_X - b_Y)(M_2-)\,\ell_t^{M_2}(Y)\right) I_{\tau_{\varepsilon,M}(Y)>t}\right].$$

By Monotone Convergence and Lemma 2.1, upon letting $\varepsilon \downarrow 0$ and $M \uparrow \infty$, we get the desired conclusion. □

Next, we prove a variant of the previous results applicable to the case $\alpha < 0$.

**Lemma 5.3.** *Let $b_1$ and $b_2$ satisfy (1.2) with the same $\beta > 0$, $M_1$ and $M_2$, but the corresponding $\alpha$'s—call them $\alpha_1$ and $\alpha_2$, respectively—are different. Suppose $b_1$ and $b_2$ are continuous on $(0,\infty)$ and for some $M \in (M_1, M_2)$, $b_1 - b_2$ restricted to $(M_1, M)$ and $(M, M_2)$ has $C^1$ extensions to $[M_1, M]$ and $[M, M_2]$, respectively. Then the function*

$$g(y) = \int_0^y (b_1 - b_2)(z)\,dz \tag{5.7}$$

*is a linear combination of convex functions with generalized second derivative $\mu$ given by*

$$\mu(A) = -\int_A \left((b_1' - b_2')\, I_{(M_1,M_2)}\right)(a)\,da, \tag{5.8}$$

*for any Borel set $A \subseteq (0,\infty)$.*

**Proof.** This is similar to the proof of Lemma 5.1, except because of the continuity of $b_1$ and $b_2$ on $(0,\infty)$, there will be no point mass term in the generalized second derivative. □

**Lemma 5.4.** *Let $x > 0$ and let*

$$\begin{cases} dX_t = dB_t + b_X(X_t)\,dt, & X_0 = x \\ dY_t = dB_t + b_Y(Y_t)\,dt, & Y_0 = x \end{cases} \tag{5.9}$$

*be such that $b_1$ and $b_2$ satisfy (1.2) with the same $\beta > 0$, $M_1$ and $M_2$, but the corresponding $\alpha$'s—call them $\alpha_1$ and $\alpha_2$, respectively—are different. Suppose $b_1$ and $b_2$ are continuous on $(0,\infty)$ and for some $M \in (M_1, M_2)$, $b_1 - b_2$ restricted to $(M_1, M)$ and $(M, M_2)$ has $C^1$ extensions to $[M_1, M]$ and $[M, M_2]$. Then for $g$ as in (5.7) and*

$$H = \left(b_1^2 - b_2^2\right) - \left(b_1' - b_2'\right) I_{(M_1,M_2)\setminus\{M\}},$$

*we have*

$$P_x(\tau_0(X) > t) = E_x\left[\exp\left(g(Y_t) - g(x) - \tfrac{1}{2}\int_0^t H(Y_s)\,ds\right) I_{\tau_0(Y)>t}\right]. \tag{5.10}$$

**Proof.** This is a simple modification of the proof of Lemma 5.1, where again, because of the continuity of $b_1$ and $b_2$ on $(0,\infty)$, there will be no local time term. □

# 6. Upper bound

The main result of this section is the following theorem, which combined with Theorem 3.1 will prove Theorem 1.1.

**Theorem 6.1.** *Let $X_t$ be as in (1.1), where $b$ is from (1.2). Then*

$$\limsup_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \le -\gamma(p,\beta),$$

*where $\gamma(p,\beta)$ is from (1.4).*

We break up the proof into the cases when $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$, in that order.

Let $\alpha > 0$. We will remove the assumptions that $b \in C^3$ and $b' \ge 0$ in Theorem 4.1.

**Lemma 6.2.** *Under the condition (1.2), if $\alpha > 0$, then the solution $X_t$ of (1.1) satisfies*

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \le -\gamma(p,\beta),$$

*where $\gamma(p,\beta)$ is from (1.4).*

The proof uses several arguments that we consider next. The basic setup is as follows. Let $0 < x_1 < x_2$, $y_1 < y_2 < 0$ and $\gamma > 0$ be given. Consider the lines

$$\tilde b_X(x) = \frac{\gamma - y_1}{x_2 - x_1}(x - x_1) + y_1, \quad x > 0 \tag{6.1}$$

and

$$\tilde b_Y(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1, \quad x > 0. \tag{6.2}$$

Then $\tilde b_X$ passes through the points $(x_1, y_1)$ and $(x_2, \gamma)$ and $\tilde b_Y$ passes through $(x_1, y_1)$ and $(x_2, y_2)$.

**Lemma 6.3.** *For $\gamma$ sufficiently large,*

$$\tilde b_X^2 - \tilde b_Y^2 - (\tilde b_X' - \tilde b_Y') \ge 0 \text{ on } [x_1, x_2].$$

**Proof.** We have

$$\tilde b_X^2 - \tilde b_Y^2 = \left(\frac{x - x_1}{x_2 - x_1}\right)^2 \left[(\gamma - y_1)^2 - (y_2 - y_1)^2\right] + \frac{2y_1(\gamma - y_2)}{x_2 - x_1}(x - x_1)$$

and $\tilde b_X^2 - \tilde b_Y^2$ takes on its minimum value $-\frac{y_1^2(\gamma - y_2)^2}{(\gamma-y_1)^2-(y_2-y_1)^2}$ at the point $x = x_1 - \frac{y_1(\gamma-y_2)(x_2-x_1)}{(\gamma-y_1)^2-(y_2-y_1)^2}$ (note that $\gamma - y_1 > y_2 - y_1 > 0$). In particular, on $[x_1, x_2]$,

$$\begin{aligned}(\tilde b_X^2 - \tilde b_Y^2)(x) + (\tilde b_X' - \tilde b_Y')(x) &\ge -\frac{y_1^2(\gamma - y_2)^2}{(\gamma - y_1)^2 - (y_2 - y_1)^2} + \frac{\gamma - y_1}{x_2 - x_1} - \frac{y_2 - y_1}{x_2 - x_1} \\ &= -\frac{y_1^2(\gamma - y_2)^2}{(\gamma - y_1)^2 - (y_2 - y_1)^2} + \frac{\gamma - y_2}{x_2 - x_1} = (\gamma - y_2)\left[-\frac{y_1^2(\gamma - y_2)}{(\gamma - y_1)^2 - (y_2 - y_1)^2} + \frac{1}{x_2 - x_1}\right] \ge 0,\end{aligned}$$

for large $\gamma$. $\square$

**Lemma 6.4.** *Let $dZ_t = dB_t + b_Z(Z_t)\,dt$, where $b_Z$ satisfies (1.2) with $b_Z = C > 0$ on $(M_1, M_2)$. Then*

$$\limsup_{t\to\infty} \log P_x(\tau_0(Z) > t) \le -\gamma(p,\beta),$$

*where $\gamma(p,\beta)$ is from (1.4).*

**Proof.** By our hypotheses,

$$b_Z(x) = \begin{cases} -\alpha x^{-q}, & x \in (0, M_1] \\ C, & x \in (M_1, M_2) \\ -\beta x^{-p}, & x \in [M_2, \infty). \end{cases} \tag{6.3}$$

In Lemma 5.2, choose $(x_2, y_2) = (M_2, -\beta M_2^{-p})$ and $(x_1, y_1) = \left(\frac{M_1}{K}, -\alpha\left(\frac{M_1}{K}\right)^{-q}\right)$ where $K > 1$ is so large that $y_1 < y_2$. Notice this tells us $y_1 < -\alpha M_1^{-q}$. Then we can choose $\gamma > 0$ so large that the conclusion of Lemma 5.2 holds for $\tilde{b}_X$ and $\tilde{b}_Y$ given by (6.1) and (6.2):

$$\tilde{b}_X^2 - \tilde{b}_Y^2 - (\tilde{b}_X' - \tilde{b}_Y') \ge 0 \text{ on } [x_1, x_2]. \tag{6.4}$$

Define

$$b_X(x) = \begin{cases} -\alpha x^{-q}, & x \in (0, x_1] \\ \tilde{b}_X(x), & x \in (x_1, x_2) = (x_1, M_2) \\ -\beta x^{-p}, & x \in [x_2, \infty) = [M_2, \infty) \end{cases} \tag{6.5}$$

and

$$b_Y(x) = \begin{cases} -\alpha x^{-q}, & x \in (0, x_1] \\ \tilde{b}_Y(x), & x \in (x_1, x_2) = (x_1, M_2) \\ -\beta x^{-p}, & x \in [x_2, \infty) = [M_2, \infty). \end{cases} \tag{6.6}$$

Then by definition of $b_X$, $b_Y$ and (6.4),

$$b_X^2 - b_Y^2 - (b_X' - b_Y')I_{(x_1,x_2)} \ge 0 \text{ on } (0,\infty). \tag{6.7}$$

Notice this continues to hold if $\gamma$ is made larger. Since $x_1 = \frac{M_1}{K} < M_1$, we can make $\gamma$ larger, if necessary, so that the line segment $\{(x, C) : M_1 < z < M_2\}$ (recall $C$ is from (6.3)) lies to the right of the line through the points $(x_1, y_1)$ and $(x_2, \gamma)$. Since $x_1 < M_1$, we have $b_Z \le b_X$ on $(0,\infty)$. By the Comparison Theorem,

$$P_x(\tau_0(Z) > t) \le P_x(\tau_0(X) > t). \tag{6.8}$$

We now apply Lemma 5.2 to $b_X$, $b_Y$ given by (6.5) and (6.6) respectively, but with $M_1, M_2$ in the lemma taken to be $x_1, x_2$. First check the hypotheses of the Lemma:

- $b_X$ and $b_Y$ are continuous on $(0,\infty)\backslash\{x_2\}$ since

$$\lim_{x\to x_1^-} b_X(x) = y_1 = \lim_{x\to x_1^+} b_X(x) \text{ and } \lim_{x\to x_1^-} b_Y(x) = y_1 = \lim_{x\to x_1^+} b_Y(x).$$

- Since each is linear on $(x_1, x_2)$, each restricted to $(x_1, x_2)$ has a $C^1$ extension to $[x_1, x_2]$.

Thus Lemma 5.2 applies and so for

$$g(z) = \int_0^x (b_X - b_Y)(y)\,dy \quad \text{and} \quad H = (b_X^2 - b_Y^2) - (b_X' - b_Y')I_{(x_1,x_2)},$$

we have

$$P_x(\tau_0(X) > t) = E_x\left[\exp\left(g(Y_t) - g(x) - \tfrac{1}{2}\int_0^t H(Y_s)\,ds - \tfrac{1}{2}(b_X - b_Y)(M_2-)\,\ell_t^{M_2}(Y)\right) I_{\tau_0(Y)>t}\right]. \tag{6.9}$$

Now by (6.7), $H \geq 0$, and by definition of $b_X$ and $b_Y$,

$$\begin{aligned}(b_X - b_Y)(M_2-) &= \lim_{x\to x_2^-}\left(\tilde{b}_X(x) - \tilde{b}_Y(x)\right) \text{ (by (6.5)–(6.6))}\\ &= \tilde{b}_X(x_2) - \tilde{b}_Y(x_2) \text{ (since } \tilde{b}_X \text{ and } \tilde{b}_Y \text{ are continuous)}\\ &= \gamma - y_2 \text{ (by (6.1)–(6.2))} > 0.\end{aligned}$$

Then (6.9) becomes

$$P_x(\tau_0(X) > t) \leq E_x\left[\exp\left(g(Y_t) - g(x)\right) I_{\tau_0(Y)>t}\right]. \tag{6.10}$$

Since $b_X = b_Y$ on $(x_1, x_2)^c$ and since $b_X - b_Y = \tilde{b}_X - \tilde{b}_Y$ is bounded on $(x_1, x_2)$, we see

$$g(z) = \int_0^z (b_X - b_Y)(y)\,dy \leq \sup_{(x_1,x_2)} (\tilde{b}_X - \tilde{b}_Y)(x_2 - x_1), \quad z > 0.$$

Hence for some positive constant $C_1$ independent of $t$, (6.10) becomes

$$P_x(\tau_0(X) > t) \leq C_1 P_x(\tau_0(Y) > t). \tag{6.11}$$

The crucial point is that $\sup_{(0,\infty)} b_Y < 0$. This holds because:

- on $(0, x_1]$, $b_Y(x) = -\alpha x^{-q} \leq -\alpha x_1^{-p}$;
- on $(x_1, x_2)$, by (6.2) and that $y_1 < y_2 < 0$, $b_Y = \tilde{b}_Y \leq y_2 < 0$;
- on $(x_2, \infty)$, $b_Y(x) = -\beta x^{-p} \leq -\beta x_2^{-p}$.

Thus Theorem 4.1 applies to the process $Y$, and we conclude

$$\limsup_{t\to\infty} \log P_x(\tau_0(Y) > t) \leq -\gamma(p, \beta),$$

where $\gamma(p, \beta)$ is from (1.4). Combined with (6.8) and (6.11), this gives us

$$\limsup_{t\to\infty} \log P_x(\tau_0(Z) > t) \leq -\gamma(p, \beta),$$

as desired. □

**Proof of Lemma 6.2.** Let $X_t$ be from (1.1), where $b$ satisfies (1.2) and $\alpha \geq 0$. Let $C = \sup_{[M_1,M_2]} |b|$. Suppose $dZ_t = dB_t + b_Z(Z_t)\,dt$, where $b_Z$ satisfies (1.2) with $b_Z = C$ on $(M_1, M_2)$. Then $b_X \leq b_Z$ and so by the Comparison Theorem, $P_x(\tau_0(X) > t) \leq P_x(\tau_0(Z) > t)$. The desired upper bound follows upon applying Lemma 6.4 to $Z$. □

Now let $\alpha < 0$.

**Lemma 6.5.** *Under the condition* (1.2)*, if* $\alpha < 0$*, then the solution* $X_t$ *of* (1.1) *satisfies*

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \le -\gamma(p,\beta),$$

*where* $\gamma(p,\beta)$ *is from* (1.4).

For the proof, we need the following special case.

**Lemma 6.6.** *Let* $\alpha < 0$*. Assume* $b \in C^3$ *satisfies condition* (1.2)*. Suppose that for some* $x_1 \in (0, M_2)$*,* $b(x_1) = 0$ *and* $b$ *is strictly decreasing on* $(0, x_1)$*. Then the solution* $X_t$ *of* (1.1) *satisfies*

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \le -\gamma(p,\beta),$$

*where* $\gamma(p,\beta)$ *is from* (1.4).

**Proof.** Define

$$b_Y(x) = \begin{cases} -b(x), & x \in (0, x_1] \\ b(x), & x \in (x_1, \infty), \end{cases} \tag{6.12}$$

and let $Y_t$ solve $dY_t = dB_t + b_Y(Y_t)\,dt, \quad Y_0 = x$.

Now $b$ and $b_Y$ are continuous on $(0,\infty)$ and $b - b_Y$ restricted to $(M_1, x_1)$ and $(x_1, M_2)$ has $C^1$ extensions to $[M_1, x_1]$ and $[x_1, M_2]$, respectively. Then by Lemma 5.4, taking $M$ there to be $x_1$, for

$$g(z) = \int_0^z (b - b_Y)(y)\,dy \quad \text{and} \quad H = \left(b^2 - b_Y^2\right) - \left(b' - b'_Y\right) I_{(M_1,M_2)\setminus\{x_1\}},$$

we have

$$P_x(\tau_0(X) > t) = E_x\left[\exp\left(g(Y_t) - g(x) - \tfrac{1}{2}\int_0^t H(Y_s)\,ds\right) I_{\tau_0(Y)>t}\right]. \tag{6.13}$$

On the other hand, $b^2 = b_Y^2$ on $(0,\infty)$ and $(b' - b'_Y)(x) = \begin{cases} 2b'(x), & x \in (M_1, x_1) \\ 0, & x \in (x_1, \infty) \end{cases} \le 0.$

In particular, $H \ge 0$ and (6.13) becomes

$$P_x(\tau_0(X) > t) \le E_x\left[\exp\left(g(Y_t) - g(x)\right) I_{\tau_0(Y)>t}\right]. \tag{6.14}$$

Since $b = b_Y$ on $(x_1, \infty)$ and $b - b_Y = 2b$ is nonnegative and integrable on $(0, x_1)$, we see $g$ is bounded. Hence for some positive constant $C_1$ independent of $t$, (6.14) becomes

$$P_x(\tau_0(X) > t) \le C_1 P_x(\tau_0(Y) > t). \tag{6.15}$$

By Lemma 6.2 applied to $Y$,

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \le \limsup_{t\to\infty} \log P_x(\tau_0(Y) > t) \le -\gamma(p,\beta),$$

where $\gamma(p,\beta)$ is from (1.4). □

**Proof of Lemma 6.5.** Let $X_t$ be from (1.1), where $b$ satisfies (1.2) and $\alpha < 0$. Let $C > \sup_{[M_1,M_2]} |b|$ and choose $\alpha_1 < \alpha$ such that $-\alpha_1 > CM_2^q$. Then choose $b_Y \in C^3$ having the following properties: for some $M_3, M_4 \in (M_2, \infty)$ with $M_3 < M_4$,

- $b_Y(x) = -\alpha_1 x^{-q}$ on $(0, M_2]$;
- $b_Y$ is strictly decreasing on $[M_2, M_3]$;
- $b_Y(M_3) = 0$;
- $b_Y = b$ on $[M_4, \infty)$.

Suppose $dY_t = dB_t + b_Y(Y_t)\,dt$. Then $b \le b_Y$ and so by the Comparison Theorem, $P_x(\tau_0(X) > t) \le P_x(\tau_0(Y) > t)$. The process $Y$ satisfies the conditions of Lemma 6.6 and the desired upper bound follows by applying that lemma to $Y$. □

Now let $\alpha = 0$ and let $Y_t$ solve $dY_t = dB_t + b_Y(Y_t)\,dt$, $Y_0 = x$, where

$$b_Y(x) = \begin{cases} x^{-q}, & x \in (0, M_1] \\ b(x), & x \in (M_1, \infty). \end{cases}$$

Then by the Comparison Theorem, $P_x(\tau_0(X) > t) \le P_x(\tau_0(Y) > t)$. Since the situation of $\alpha < 0$ holds for $b_Y$, that case implies

$$\limsup_{t\to\infty} \log P_x(\tau_0(X) > t) \le \limsup_{t\to\infty} \log P_x(\tau_0(Y) > t) \le -\gamma(p,\beta),$$

once again. □

## 7. Proof of Theorem 1.2

**Lemma 7.1.** *Let $\delta > 0$ be so small that $0 < p - \delta < p + \delta < 1$ and $0 < q - \delta < q + \delta < 1$. By decreasing $M_1$ and increasing $M_2$ if necessary, we have*

$$x^{-\delta} \le \ell_2(x) \le x^{\delta}, \quad \textit{for} \quad x \ge M_2, \quad \textit{and} \quad x^{\delta} \le \ell_1(x) \le x^{-\delta}, \quad \textit{for} \quad x \le M_1.$$

**Proof.** By making $M_2$ bigger if necessary, by Theorem 3.2 (a) in Supplement [9], we get

$$M_2^{-\delta}\ell_2(M_2) \le 1/2 < 2 \le M_2^{\delta}\ell_2(M_2). \tag{7.1}$$

Then by Theorem 3.1 (a) in Supplement [9], again by making $M_2$ bigger if necessary, we have for $x \ge M_2$,

$$\ell_2(x) \le 2\max\left((x/M_2)^{\delta}, (x/M_2)^{-\delta}\right)\ell_2(M_2) = 2(x/M_2)^{\delta}\ell_2(M_2) = 2x^{\delta}M_2^{-\delta}\ell_2(M_2) \le 2x^{\delta}(1/2) = x^{\delta},$$

and

$$\begin{aligned}\ell_2(M_2) &\le 2\max\left((M_2/x)^{\delta}, (M_2/x)^{-\delta}\right)\ell_2(x) = 2(x/M_2)^{\delta}\ell_2(x) \le (M_2^{\delta}\ell_2(M_2))(x/M_2)^{\delta}\ell_2(x)\\ &= x^{\delta}\ell_2(M_2)\ell_2(x).\end{aligned}$$

Rearranging the latter, we have $x^{-\delta} \le \ell_2(x)$, so combined we get that if $x \ge M_2$ then $x^{-\delta} \le \ell_2(x) \le x^{\delta}$.

Similarly, by making $M_1$ smaller if necessary, by Theorem 3.2 (b) and Theorem 3.1 (b) of Suplement [9], we have that if $x \le M_1$, then $x^{\delta} \le \ell_1(x) \le x^{-\delta}$. □

**Proof of Theorem 1.2.** Let $\varepsilon > 0$ be small so that $0 < \beta_0 - \varepsilon < \beta_0 + \varepsilon < 1$, and if necessary, decrease $M_1$ and increase $M_2$ so that Lemma 7.1 continues to apply. We have $\alpha_0 = \sup_{(0,M_1]} |\alpha| < \infty$, and if $x \geq M_2$ then $0 < \beta_0 - \varepsilon < \beta(x) < \beta_0 + \varepsilon < 1$.

Define

$$\widetilde{b}(x) = \begin{cases} \alpha_0 x^{q-\delta}, & 0 < x \leq M_1 \\ b(x), & M_1 < x < M_2 \\ -(\beta_0 + \varepsilon) x^{p+\delta}, & M_2 \leq x, \end{cases} \quad \text{and} \quad \overline{b}(x) = \begin{cases} -\alpha_0 x^{q-\delta}, & 0 < x \leq M_1 \\ b(x), & M_1 < x < M_2 \\ -(\beta_0 - \varepsilon) x^{p-\delta}, & M_2 \leq x. \end{cases}$$

Let $\widetilde{X}_t$ and $\overline{X}_t$ be the solutions of the equations

$$d\widetilde{X}_t = dB_t + \widetilde{b}(\widetilde{X}_t)\, dt, \quad \widetilde{X}_0 = x, \quad \text{and} \quad d\overline{X}_t = dB_t + \overline{b}(\overline{X}_t)\, dt, \quad \overline{X}_0 = x.$$

Since $\overline{b} \leq b \leq \widetilde{b}$, by the Comparison Theorem we have

$$P_x(\tau_0(\overline{X}) > t) \leq P_x(\tau_0(X) > t) \leq P_x(\tau_0(\widetilde{X}) > t).$$

But Theorem 1.1 applies to $\widetilde{X}_t$ and $\overline{X}_t$ and so we get

$$-\gamma(p - \delta, \beta_0 - \varepsilon) \leq \liminf_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t)$$

$$\leq \limsup_{t\to\infty} t^{-(1-p)/(1+p)} \log P_x(\tau_0(X) > t) \leq -\gamma(p + \delta, \beta_0 + \varepsilon).$$

Let $\varepsilon$ and $\delta$ go to 0 to get the desired conclusion. □

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### Supplementary Material

**Supplement to "Subexponential estimates for the first hitting time of a Brownian motion with singular drift"**
The supplementary material [9] contains auxiliary results that help the reader with some technical details. In Section 1, a summary of $h$-transform results from Pinsky [23] leads to an important identity that is used in Lemma 2.5. In Section 2, we summarize the main variational results from DeBlassie and Smits [10] needed in the proof of Theorems 3.2 and 4.1. In the proof of Theorem 1.2 we used some bounds and representations of slowly varying functions from Bingham, Goldie and Teugels [5], which are summarized for convenience in Section 3.

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