Large deviation principle for additive functionals of semi-Markov processes

ADINA OPRISAN*

Department of Mathematical Sciences, New Mexico State University

A large deviation principle (LDP) for a class of additive functionals of semi-Markov processes and their associated Markov renewal processes is studied via an almost sure functional central limit theorem . The rate function corresponding to the deviations from the paths of the corresponding empirical processes with logarithmic averaging is determined as a relative entropy with respect to the Wiener measure on $\mathcal{D}[0,\infty)$. *A martingale decomposition for additive functionals of Markov renewal processes is employed.*

Keywords: semi-Markov processes; renewal processes; almost sure functional central limit theorem; large deviations, martingale decomposition.

AMS Subject Classification: 60B10, 60F10, 60F17, 60G57.

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete probability space equipped with the right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $(E, \mathcal{E})$ be a complete separable metric space. Let $\{X(t) : t \geq 0\}$ be an $(E, \mathcal{E})$-valued progressively measurable, time-homogeneous semi-Markov process with semi-Markov kernel $Q(x, A\times\Gamma)$, $x \in E$, $A \in \mathcal{E}$, $\Gamma \in \mathcal{B}_+$ on $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$ ($\mathcal{B}_+$ is the Borel σ-algebra of $\mathbb{R}_+$). Specifically, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider a jump type Markov process, with values in $(E, \mathcal{E})$. If $0 = \tau_0 < \tau_1 < ....$ are the jump times, one can define a discrete-time process $\{X_n,\, n = 0, 1, ...\}$ by $X_n := X(\tau_n)$. This is a Markov chain with the state space E and the transition probability kernel $p(x, dy) := Q(x, dy \times [0,\infty))$. Given a probability measure μ on $(E, \mathcal{E})$, one can define the probability measures $\mathbb{P}_\mu$ as:

$$\mathbb{P}_\mu(A) = \mu\mathbb{P}(A) = \int_E \mu(dx)\, p(x, A), \quad x \in E,\, A \in \mathcal{F}.$$

Let P be the transition probability operator corresponding to the transition probability $p(x, A)$,

$$P\varphi(x) := \mathbb{E}[\varphi(X_{n+1}) \,|\, X_n = x] = \int_E p(x, dy)\varphi(y),$$

*Address correspondence to Adina Oprisan, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM; E-mail: aoprisan@nmsu.edu

and P^n the n-step transition operator corresponding to the n-step transition probability $p^n(x, A)$.

The stochastic process $\{(X_n, \tau_n),\, n \geq 0\}$ is called the embedded Markov renewal process with renewal times τ_n and

$$\mathbb{P}(X_{n+1} \in A,\, \tau_{n+1} - \tau_n \in \Gamma \,|\, X_n = x) = Q(x, A \times \Gamma)$$

for any $n \geq 0$, $A \in \mathcal{E}$, and $\Gamma \in \mathcal{B}_+$.

Let $N(t) = \max\{n : \tau_n \leq t\}$ be the point process that counts the jumps of X in the time interval $(0, t]$, and the semi-Markov process $\{X(t) : t \geq 0\}$ defined by $X(t) := X_{N(t)}$. For $n \geq 1$, define the inter-jumps times $\theta_n = \tau_n - \tau_{n-1}$, $n \geq 1$. The random variable θ_n is also called the sojourn time in the state X_n, and for any given $\{X_n,\, n \geq 0\}$ the random variables $\{\theta_n,\, n \geq 0\}$ are mutually independent. Denote by θ_x, $x \in E$, the sojourn time in the state x, and let $F_x(t) = \mathbb{P}\{\theta_{n+1} \leq t \,|\, X_n = x\} = Q(x, E \times [0, t])$ be the sojourn distribution in the state x. The mean sojourn time in the state x, $m(x)$, is

$$m(x) := \int_0^\infty \mathbb{P}(\theta_{n+1} > t \,|\, X_n = x)\, dt = \int_0^\infty \bar{F}_x(t)\, dt, \quad \bar{F}_x(t) = 1 - F_x(t).$$

Assume that the stochastic kernel $p(x, A)$ induces an ergodic Markov chain with stationary distribution ν, such that the mean sojourn time is

$$m := \int_E \nu(dx)\, m(x) < \infty.$$

The two-component process $\{(X_n, \theta_{n+1}),\, n \geq 0\}$ taking values in $E \times [0, \infty)$ is a Markov process, and in the literature it is also called Markov renewal process. Its transition probabilities are given in terms of the semi-Markov kernel

$$Q(x, A \times \Gamma) = \mathbb{P}(X_{n+1} \in A, \theta_{n+2} \in \Gamma \,|\, X_n = x). \tag{1.1}$$

The transition operator of the Markov renewal process will be then defined as:

$$Qf(x) := \int_{E \times \mathbb{R}+} Q(x, dy \times ds) f(y, s), \tag{1.2}$$

for $f : E \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable. Any measure μ on $(E, \mathcal{E})$ induces a measure μQ on $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$ by

$$\mu Q(A \times \Gamma) = \int_E \mu(dx) Q(x, A \times \Gamma).$$

The n^{th} convolution of the semi-Markov kernel Q defined as,

$$Q^{(n)}(x, A \times \Gamma) = \int_{E \times \mathbb{R}_+} Q(x, dy \times ds) Q^{(n-1)}(y, A \times (\Gamma - s)), \quad n \geq 2, \tag{1.3}$$

gives the n-step transition probability of the Markov renewal process,

$$Q^{(n)}(x, A, \Gamma) = \mathbb{P}(X_n \in A, \theta_{n+1} \in \Gamma \,|\, X_0 = x).$$

Thus $Q^{(n)}(x, A, t) = P^n(x, A)F_x(t)$.

Also, the n-step transition operator of the Markov renewal process is defined as

$$Q^n f(x) := \int_{E\times\mathbb{R}+} Q^{(n)}(x, dy \times ds) f(y, s), \tag{1.4}$$

for $f : E \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable. Let $\mu g := \int_E \mu(dx)g(x)$ for any g real-valued measurable function on E and define the functionals

$$\mu Q f := \int_{E\times E\times\mathbb{R}_+} \mu(dx)Q(x, dy \times ds)f(y, s)$$

and

$$\mu Q^n f := \int_{E\times E\times\mathbb{R}_+} \mu(dx)Q^{(n)}(x, dy \times ds)f(y, s).$$

Throughout the paper, derivations are made under the following assumptions, unless otherwise specified.

A1. The semi-Markov process X is regular:

$$(\forall)\, x \in E,\ (\forall)\, t \geq 0,\ \mathbb{P}_x(N(t) < \infty) = 1;$$

A2. The Markov chain $\{X_n,\ n \geq 0\}$ is ergodic and Harris recurrent, with stationary distribution ν. Recall that a Markov chain is Harris recurrent if it is irreducible and for any set $A \in \mathcal{E}$ such that $\nu(A) > 0$, and for any intial distribution μ, we have $\mathbb{P}_\mu(x_n \in A\, i.o.) = 1$;

A3. The mean sojourn time in a state $x \in E$ is uniformly bounded: $a \leq \sup_{x\in E} m(x) \leq b,\ a, b > 0$;

A4. The family of sojourn times $\{\theta_x,\ x \in E\}$ is uniformly integrable, i.e. as $N \to \infty$,

$$\sup_{x\in E} \int_N^\infty \mathbb{P}(\theta_{n+1} > t \,|\, X_n = x)\, dt \to 0.$$

Below we state two important results regarding Markov renewal processes. For more explanations we refer to [8, 16] and references therein.

Lemma 1.1 *The Markov renewal processes $\{(X_n,\ \tau_n),\ n \geq 0\}$ and $\{(X_n,\ \theta_{n+1}),\ n \geq 0\}$ have the same stationary measure that is induced by the stationary measure of the embedded Markov chain: if ν is the stationary measure of the embedded Markov chain $\{X_n,\ n \geq 0\}$, then $\tilde{\nu} := \nu F$ defined by $\tilde{\nu}(dy \times ds) = \nu(dy)F_y(ds)$ is the stationary measure for the Markov renewal processes.*

For fixed $t \in \mathbb{R}_+$, define $Y(t) := t - \tau_{N(t)}$ as the amount of time the process $X(t)$ is at the current state after the last jump.

Lemma 1.2 *The process* $\{(X(t), Y(t)),\, t \geq 0\}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a jointly Markov process with stationary distribution on* $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$,

$$\tilde{\pi}(A \times \Gamma) = \frac{1}{m} \int_A \nu(dx) \int_\Gamma (1 - F_x(u))\, du.$$

The marginal law $\pi(A) = \tilde{\pi}(A \times \mathbb{R}_+)$ *on* $(E, \mathcal{E})$, *is the stationary probability measure for the semi-Markov process* $\{X(t),\, t \geq 0\}$ *and*

$$\pi(A) = \frac{1}{m} \int \nu(dx) m(x) = \lim_{t \to \infty} \mathbb{P}(X(t) \in A | X(0) = x) \quad \textit{for} \quad (\forall)\, x \in E. \tag{1.5}$$

2. Additive functionals of semi-Markov processes

Functional limit theorems for additive functionals of Markov and semi-Markov processes have been extensively studied in the literature. We refer to [9] for a functional central limit theorem for semi-Markov processes in a discrete state space and [15] for one in a more general context. An invariance principle for additive functionals of Markov processes is due to R.N. Bhattacharya [2]. Functional central limit theorems, almost sure central limit theorems and large deviations for additive functionals of Markov processes have been discussed in [12, 18]. In this section we will extend our previous results on additive functionals of Markov processes to the semi-Markov setup. We will consider a class of additive functionals of ergodic semi-Markov processes and prove that their associated Markov renewal processes satisfy a martingale decomposition, which is a key result that leads us to a functional central limit theorem, an almost sure central limit theorem and ultimately, to a large deviation principle for additive functionals of semi-Markov processes.

Let $\{X_t,\, t \geq 0\}$ be a semi-Markov process with stationary probability measure π and $f : E \to \mathbb{R}_+$ a Borel measurable function. Consider the additive functional

$$W_t := \int_0^t f(X(s))\, ds, \tag{2.1}$$

that can be viewed as the reward earned over the interval $[0, t]$ in a game where the reward at time s is $f(x)$, if $X(s) = x$. This is equivalent to,

$$W_t = \sum_{k=1}^{N(t)} f(X_{k-1})\theta_k + (t - \tau_{N(t)}) f(X_{N(t)}).$$

Lemma 2.1 *Let* $\{X_t,\, t \geq 0\}$ *be an ergodic semi-Markov with ergodic distribution* π *and* $\{X_n,\, n \geq 0\}$ *its embedding Markov chain with stationary distribution* ν. *Assume that the function* $f \in L^2(\pi)$ *satisfies the following conditions:*

(i) $\int_E f\, d\pi = 0$,

(ii) there exists $0 < c < \infty$ *such that* $d\,\mu P^k \leq c\, d\nu$ *for any* $k \in \mathbb{N}$ *and* $\int_{\{x : f^2(x) > n\}} f^2(x) \nu(dx) \leq \phi(n)$ *where*

$$\phi : \mathbb{R}_+ \to \mathbb{R}_+, \textit{ such that } \lim_{x \to \infty} \phi(x) = 0.$$

Then, the random processes

$$W_t^n = \frac{1}{\sigma\sqrt{n}} \int_0^{nt} f(X_u)\, du \quad \textit{and} \quad \tilde{W}_t^n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k,$$

have the same limiting distribution.

Proof: We will show that for any $T > 0$, $\sup_{0\le t<T} |W_t^n - \tilde{W}_t^n|$, converges in probability to zero. We have:

$$\begin{gathered}
\mathbb{P}(\sup_{0\le t\le T} |W_t^n - \tilde{W}_t^n| > \varepsilon) \le \\
\mathbb{P}(\sup_{0\le t\le T} |(nt - \tau_{N(nt)}) f(X_{N(nt)})| > \varepsilon\sigma\sqrt{n}) \le \\
\mathbb{P}(\sup_{0\le t\le T} \theta_{N(nt)+1} |f(X_{N(nt)})| > \varepsilon\sigma\sqrt{n}) \le \\
\mathbb{P}(\sup_{0\le t\le T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{n}) + \mathbb{P}(\sup_{0\le t\le T} |f(X_{N(nt)})| > \sqrt[4]{n}).
\end{gathered} \tag{2.2}$$

The first term in (2.2) can be upper bounded successively:

$$\begin{gathered}
\mathbb{P}(\sup_{0\le t\le T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{n}) \le \\
\mathbb{P}(\sup_{0\le t\le T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{n},\, N(nT) \le N) + \mathbb{P}(N(nT) > N) \le \\
P(\sup_{0\le k\le N} \theta_{k+1} > \varepsilon\sigma\sqrt[4]{n}) + \mathbb{P}(N(nT) > N) \le \\
\textstyle\sum_{k=0}^{N} \mathbb{P}(\theta_{k+1} > \varepsilon\sigma\sqrt[4]{n}) + \mathbb{P}(N(nT) > N) \le \\
(N+1) sup_{x\in E} \int_{\varepsilon\sigma\sqrt[4]{n}}^{\infty} \bar{F}_x(t)\, dt + \mathbb{P}(N(nT) > N), \text{ for any } N > 0.
\end{gathered}$$

As $n \to \infty$, the first term goes to zero due to the uniform integrability condition while the second term goes to zero as $N \to \infty$, due to the regularity condition [$A1$].

Similarly, the second term in (2.2) can be estimated as follows:

$$\begin{gathered}
\mathbb{P}(\sup_{0\le t\le T} |f(X_{N(nt)})| > \sqrt[4]{n}) \le \\
\mathbb{P}(\sup_{0\le t\le T} |f(X_{N(nt)})| > \sqrt[4]{n},\, N(nT) \le N) + \mathbb{P}(N(nT) > N) \le \\
\mathbb{P}(\sup_{0\le k\le N} |f(X_k)| > \sqrt[4]{n}) + \mathbb{P}(N(nT) > N) \le \\
\textstyle\sum_{k=0}^{N} \mathbb{E}|f(X_k)| \mathbb{1}_{\{|f(X_k)| > \sqrt[4]{n}\}} + \mathbb{P}(N(nT) > N) \le \\
\textstyle\sum_{k=0}^{N} \int_{\{f^2(x) > \sqrt{n}\}} |f(x)| d\mu P_k(dx) + \mathbb{P}(N(nT) > N) \le \\
c \textstyle\sum_{k=0}^{N} \int_{\{f^2(x) > \sqrt{n}\}} |f(x)| d\nu + \mathbb{P}(N(nT) > N) \le \\
cN\phi(n) + \mathbb{P}(N(nT) > N).
\end{gathered}$$

As $n \to \infty$ and $N \to \infty$,the above term goes to zero because of condition (ii) and [A1].

Since $A_T = \{\sup_{0\le t\le T} |W_t^n - \tilde{W}_t^n| > \varepsilon\}$ is increasing, by taking the limit as $t \to \infty$, we get that $\sup_{0\le t<\infty} |W_t^n - \tilde{W}_t^n|$ converges in probability to zero. ■

Martingale decompositions for additive functionals of discrete and continuous Markov processes have been established in [12, 18]. The next result shows that a similar martingale decomposition can be obtained for Markov renewal processes.

Theorem 2.2 *Let $\{X(t),\, t \geq 0\}$ be an ergodic semi-Markov process with initial distribution μ and unique invariant measureπ defined in (1.5). Let $\{X_n,\, n \geq 0\}$ be its embedding Markov chain with stationary distribution ν and $f \in L^2(E, \pi)$ satisfying the following conditions:*

(i) $\int_E f(x)\pi(dx) = 0$;

(ii) $\|P^k f\|_{L^2(\nu)} \leq \rho^k \|f\|_{L^2(\nu)}$, *for some* $0 < \rho < 1$, $k \in \mathbb{N}$;

(iii) there exists $0 < c < \infty$ *such that* $d\mu P^k \leq c\, d\nu$ *for any* $k \in \mathbb{N}$ *and* $\int_{\{x: f^2(x) > n\}} f^2(x)\nu(dx) \leq \exp(-\varphi(n))$ *for* n *large, with* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\lim_{x\to\infty} \frac{\varphi(x)}{\log x} = \infty$.

(iv) $|P^k f(x)| \leq dn$, *whenever* $|f(x)| \leq n$ *for some* $0 < d < \infty$ *and* n *sufficiently large.*

Then, the additive functional of the Markov renewal process satisfies the martingale decomposition:

$$S_n(f) := \sum_{k=1}^{n} f(X_{k-1})\theta_k = M_n + R_n \tag{2.3}$$

where M_n *is a mean zero martingale with respect to the filtration* $\mathcal{F}_n = \sigma\{X_k,\, 0 \leq k \leq n\}$, *and the remainder* R_n *converges to zero in probability. Moreover,*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left\{ \frac{\sup_{1\leq k\leq n} R_k^2}{n} > \varepsilon \right\} = -\infty. \tag{2.4}$$

Proof: Since $\{(X_{n-1}, \theta_n),\, n \geq 1\}$ is the corresponding renewal Markov process associated to the semi-Markov process, according to Lemma 1.1, it is stationary with probability invariant measure $\tilde{\nu} = \nu \cdot F$, $\tilde{\nu}(dy \times ds) = \nu(dy)F_y(ds)$. The transition probabilities and transition operators are defined in (1.1) and (1.2) respectively.

Define the measurable function $g : E \times \mathbb{R}_+ \to \mathbb{R}_+$ as $g(y, s) = f(y)s$, and let $S_n(g) = \sum_{k=1}^{n} g(X_{k-1}, \theta_k)$. Since $f \in L_0^2(\pi)$, we get that $g \in L_0^2(\tilde{\nu})$:

$$\begin{aligned} \int_{E\times\mathbb{R}_+} g(y,s)d\tilde{\nu}(dy \times ds) &= \int_E f(y)\nu(dy) \int_{\mathbb{R}_+} sF_y(ds) \\ &= \int_E f(y)m(y)\nu(dy) = m \int_E f(y)\pi(dy) = 0. \end{aligned}$$

Following the pattern of Theorem 3.1 in [12], we find the unique solution of the Poisson equation $(I - Q)u = g$. Since $I - Q$ is not invertible, consider the Poisson equation $((1 + \epsilon)I - Q)u_\epsilon = g$, $\epsilon > 0$, with unique solution u_ϵ,

$$u_\epsilon = ((1 + \epsilon)I - Q)^{-1} g = \frac{1}{1 + \epsilon} \sum_{k=1}^{\infty} \frac{Q^{k-1} g}{(1 + \epsilon)^{k-1}},$$

where $Q^n g$ is the n-step transition operator of the Markov renewal process defined in (1.4). The series is convergent in $L^2(\nu)$, since

$$\begin{aligned}
\|Q^n g\|^2_{L^2(\nu)} &= \mathbb{E}_\nu (Q^n g(x,s))^2 = \mathbb{E}_\nu \left(\int_{E\times\mathbb{R}_+} Q^{(n)}(x, dy \times ds) f(y) s \right)^2 \\
&= \mathbb{E}_\nu \left(\int_E P^n(x,dy) f(y) m(y) \right)^2 \\
&\le b\|P^n f\|_{L^2(\nu)} \le b\rho^k \|f\|_{L^2(\nu)},
\end{aligned}$$

due to the assumption [A3] and condition (ii). Thus, for $x \in E$,

$$u(x) := \lim_{\epsilon \to 0} u_\epsilon(x) = \sum_{n=0}^{\infty} P^n g(x),$$

and the martingale decomposition follows:

$$S_n(g) = \sum_{k=1}^{n} g(X_{k-1}, \theta_k) = M_n + R_n,$$

where $\{M_n,\, n \ge 0\}$ is a mean zero martingale with respect to the filtration $\mathcal{F}_n$, defined as

$$M_n = \sum_{k=1}^{n} (u(X_k) - Qu(X_{k-1})), \tag{2.5}$$

and the remainder term is defined as

$$R_n = u(X_0) - u(X_n), \quad u(X_n) = \sum_{n=0}^{\infty} Q^n g(X_n).$$

In order to prove (2.4), note that for each $k \in \mathbb{N}$, and for $C > 1$,

$$\begin{aligned}
\mathbb{P}(u^2(X_k) > Cn) &\le \mathbb{P}\left(\left| \sum_{i=0}^{\sqrt[4]{n}} Q^i g(X_k) \right| > \frac{\sqrt{Cn}}{2} \right) \\
&+ \mathbb{P}\left(\left| \sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k) \right| > \frac{\sqrt{Cn}}{2} \right)
\end{aligned}$$

The second term with $n \ge 4$ satisfies:

$$\mathbb{P}\left(\left| \sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k) \right| > \frac{\sqrt{Cn}}{2} \right) \le \mathbb{E} \left| \sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k) \right| =$$

$$\left\| \sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k) \right\|_{L^1(\Omega)} \le \left\| \sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k) \right\|_{L^2(\Omega)} \le$$

$$\sum_{i=\sqrt[4]{n}+1}^{\infty} \|Q^i g(X_k)\|_{L^2(\Omega)}$$

Based on the assumption [A3] and conditions (ii) and (iii) we have:

$$\|Q^i g(X_k)\|^2_{L^2(\Omega)} = \mathbb{E}(Q^i g(X_k))^2 = \mathbb{E}\left(\int_{E\times\mathbb{R}+} g(y,s) Q^{(i)}(X_k, dy\times ds)\right)^2 =$$
$$\mathbb{E}\left(\int_{E\times\mathbb{R}+} f(y) s P^i(X_k, dy) F_y(ds)\right)^2 \leq \left(\sup_{y\in\mathbb{R}+} m(y)\right) \mathbb{E}\left(\int_E f(y) P^i(X_k, dy)\right)^2$$
$$\leq b \int \left(\int f(y) P^i(x,dy)\right)^2 \mu P^k(dx) \leq b\cdot c\|P^i f(x)\|^2_{L^2(\nu)} \leq C_1 \rho^{2i} \|f(x)\|^2_{L^2(\nu)}.$$

Therefore, for n sufficiently large,

$$\mathbb{P}\left(\left|\sum_{i=\sqrt[4]{n}+1}^{\infty} Q^i g(X_k)\right| > \frac{\sqrt{Cn}}{2}\right) \leq \sqrt{C_1}\frac{\rho^{\sqrt[4]{n}+1}}{1-\rho}\|f\|_{L^2(\nu)} \leq A\exp(-B\sqrt[4]{n}),$$

for some positive constants A and B.

On the other hand, note that $Q^i g(X_k) = \int P^i(X_k, dy) f(y) m(y)$, therefore we get the following estimation:

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i=0}^{\sqrt[4]{n}} Q^i g(X_k)\right| > \frac{\sqrt{Cn}}{2}\right) &\leq \sum_{i=0}^{\sqrt[4]{n}} \mathbb{P}\left(|Q^i g(X_k)|^2 > \frac{Cn}{4}\right) \\ &\leq \sum_{i=0}^{\sqrt[4]{n}} \mathbb{P}\left(|P^i f(X_k)|^2 > \frac{C}{a}\frac{n}{4}\right).\end{aligned}$$

Take $\Omega = \{f^2(X_k) > \frac{n}{4}\} \cup \{f^2(X_k) \leq \frac{n}{4}\}$, and get

$$\begin{aligned}\mathbb{P}\left(|P^i f(X_k)|^2 > \frac{Cn}{4a}\right) &\leq \mathbb{P}(f^2(X_k) > \frac{n}{4}) \\ &+ \mathbb{P}\left(\{f^2(X_k) \leq \frac{n}{4}\} \cap \{|P^i f(X_k)|^2 > \frac{Cn}{4a}\}\right).\end{aligned}$$

The second term cancels out because of (iv), while the first term

$$\mathbb{P}\left(f^2(X_k) > \frac{n}{4}\right) = \mu\mathbb{P}^k\left(f^2 > \frac{n}{4}\right) \leq \exp\left(-\varphi\left(\frac{n}{4}\right)\right),$$

due to (iii). Combining the above estimates,

$$\begin{aligned}\mathbb{P}(\max_{1\leq k\leq n} |R_k| > \sqrt{\varepsilon C n}) &\leq n\mathbb{P}(|R_k| > \sqrt{\varepsilon Cn}) \leq 2n\mathbb{P}(|u(X_k)| > \frac{\sqrt{\varepsilon Cn}}{2}) \\ &\leq 2n\left[A\exp\left(-B\sqrt[4]{n}\right) + \sqrt[4]{n}\exp(-\varphi\left(\frac{n}{4}\right))\right],\end{aligned}$$

and (2.4) follows.

■

Remark 2.3 *In [12] we found that the conditions of the martingale decomposition are satisfied for the following classes of processes: finite state irreducible, aperiodic Markov chains; uniformly ergodic Markov chains in the case of bounded functionals; Markov chains obtained by quantization of continuous Markov processes that are symmetric in L^2, including the Ornstein-Uhlenbeck process.*

The Anscombe-Donsker invariance principle for Markov chains is stated next and will be used for proving the functional central limit theorem of the semi-Markov processes.

Theorem 2.4 *(Anscombe-Donsker Invariance principle for Markov chains) Let $\{X_n,\, n \geq 1\}$ be an ergodic Markov chain with stationary distribution π such that $\mathbb{E}_\pi(X_1) = 0$ and $Var_\pi(X_1) = \sigma^2 < \infty$, and $S_n = \sum_{k=1}^n X_k$, $n \geq 0$, $S_0 = 0$. Define*

$$X_n(t,\omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega) \quad (t \geq 0).$$

Suppose that $\{N(t),\, t \geq 0\}$is a nondecreasing, right continuous family of positive, integer valued random variables such

$$\frac{N(t)}{t} \xrightarrow{a.s.} \theta \quad (0 < \theta < \infty) \quad as \quad t \to \infty.$$

Define $Y_n(t,\omega) = \frac{1}{\sigma\sqrt{n}} S_{N(nt,\omega)}(\omega)$. Then, if $X_n \Rightarrow W$ then $\theta^{-1/2} Y_n \Rightarrow W$.

Theorem 2.5 *Let $\{X(t),\, t \geq 0\}$ be a stationary ergodic semi-Markov process with invariant distribution π and $f \in L^2(E,\pi)$ satisfying the assumptions of Theorem 2.2, and $\{X_n, n \geq 1\}$ be the embedded Markov chain. Then, the process $W^{(n)}$ defined by*

$$W_t^{(n)} := \frac{1}{\sigma\sqrt{nm}} \int_0^{nt} f(X_s)\, ds, \tag{2.6}$$

converges weakly to the standard Wiener process W on $\mathcal{D}([0,\infty), E)$, the space of càdlàg functions.

Proof: According to the martingale decomposition from Theorem 2.2,

$$S_t^n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} f(X_{k-1})\theta_k = \frac{1}{\sigma\sqrt{n}} M_t^{(n)} + \frac{1}{\sigma\sqrt{n}} R_t^n = M^n(t) + \frac{1}{\sigma\sqrt{n}} R_t^n, \tag{2.7}$$

where $M_t^{(n)}$ is a mean zero martingale and R_t^n converges in probability to zero. Let $A_n = \{\omega : \sup_{t\in[0,\infty)} \frac{|R_t^n(\omega)|}{\sigma\sqrt{n}} \geq \varepsilon\}$. From the relation (2.4) we get that $\mathbb{P}(A_n) \leq n^{-a_n}$, where a_n is a sequence converging to infinity. Borel-Cantelli lemma implies that A_n converges to zero $\mathbb{P}$-a.s., and based on the martingale invariance principle ([3]), we get the weak convergence of S_t^n to the Wiener measure on $\mathcal{D}([0,\infty), E)$.

Since $\sup_{0\leq t<\infty} \left| \frac{N(nt)}{n} - \frac{t}{m} \right| \to 0$ as $n \to \infty$, Anscombe invariance principle implies that $\tilde{W}_t^n = \frac{1}{\sigma\sqrt{nm}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k$ converges weakly to the Wiener process W. The conclusion follows according to Lemma 2.1.

■

Remark 2.6 *If $f : E \to \mathbb{R}$ is bounded and the underlying Markov process is uniformly ergodic, then the conditions of Theorem 2.2 are clearly satisfied and the result obtained is the same as in [15]. In that paper, σ is calculated explicitly in terms of the potential operator:*

$$\sigma^2 = \frac{1}{m} \int_E \nu(dx)[2m^2(x)f^2(x) - m_2(x)f^2(x) - 2m(x)f(x)R_{00}f(x)m(x)],$$

with $R_{00} = (I - P + P^\infty)^{-1} - P^\infty$ the potential operator of the Markov chain $\{X_n\}$.

Next, we consider empirical distributions associated with the additive functionals and prove an almost sure functional central limit theorem. The empirical measures associated with a sequence of random variables $X_n, n \geq 0$ are defined as

$$Q_n = \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{X_k}, \quad \text{where} \quad L(n) = \sum_{k=1}^{n} \frac{1}{k}$$

One may replace $L(n)$ by $\log n$. Statements regarding the convergence of the empirical distributions to a limit distribution with probability one are referred to as almost sure limit theorems. Whenever the limit distribution is Gaussian, the convergence is referred to as an almost sure central limit theorem (ASCLT). Almost sure central limit theorems for sequences of independent and identically distributed random variables were established by Brosamler [4], Shatte [19], and Lacey and Phillip [13] . Various versions of almost sure central limits theorems for martingales can be found in [17, 6, 5, 14, 1].

Theorem 2.7 *Under the conditions of Theorem 2.5, the sequence of empirical measures with logarithmic averaging associated to the additive functional $W_t^{(n)}$,*

$$\mu_n^W(\omega) := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{W^{(k)}(\omega)}, \tag{2.8}$$

satisfies the following almost sure convergence:

$$lim_{n\to\infty} \mu_n^W = \boldsymbol{W}, \quad a.s., \tag{2.9}$$

where $\boldsymbol{W}$ is the Wiener measure on $\mathcal{D}[0, \infty)$.

Proof: The equation (2.9) is equivalent to

$$lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f(W^{(n)}) = \int f d\mathbf{W}$$

for any bounded Lipschitz function h defined on $\mathcal{D}[0, \infty)$. According to Lemma 2.1, the processes W_t^n and $\tilde{W}_t^n$ have the same limiting distribution and, due to Theorem 2.4, it remains to prove that

$$lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f(S^n) = \int f d\mathbf{W}$$

where $S^n(f) = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^n f(X_{k-1})\theta_k$. The simplified version of the invariance principle with logarithmic averaging for martingales proved by F. Maaouia in [17], is

$$lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f(M^n) = \int f d\mathbf{W}.$$

Thus, it suffices to prove that

$$\lim_{n\to\infty} \|S^n - M^n\|_\rho = 0, \quad \mathbb{P} - a.e.$$

where ρ is the Skorokhod metric on $\mathcal{D}[0,\infty]$. The uniform metric on $\mathcal{D}[0,\infty]$,

$$\|x - y\| = \sup_{t\in[0,\infty)} |x(t) - y(t)|$$

is finer than ρ, therefore it is enough to see that $\lim_{n\to\infty} \|S^n - M^n\| = 0$, $\mathbb{P}$-a.s.. This follows from same argument used in the proof of Theorem 2.5. That is, if we take $A_n = \{\omega : sup_{t\in[0,\infty)} \frac{|R_t^{(n)}(\omega)|}{\sigma\sqrt{n}} \geq \varepsilon\}$, then A_n converges to 0 $\mathbb{P}$-a.s.. Finally, we conclude that almost surely, μ_n^W converges to the Wiener measure $\mathbf{W}$ on $\mathcal{D}[0,\infty)$.

■

3. Large deviation principle

In this section our goal is to study the large deviation principle for the sequence of random measures μ_n^W on $\mathcal{D}([0,\infty), E)$ defined in (2.8) and satisfying the almost sure central limit theorem (Theorem 2.7). This is a level 3 LDP with a rate function similar to the Donsker-Varadhan rate function for Brownian motion, obtained by the contraction principle from the Ornstein-Uhlenbeck process ([7]).

We studied large deviations for additive functionals of Markov processes in [18, 12]. Starting with the Donsker-Varadhan rate function for Brownian motion given in terms of the relative entropy on the underlying space $\mathcal{C}[0,\infty)$, M.K. Heck introduced a finer topology ([10]) under which the rate function was extended to some subspaces of $\mathcal{C}[0,\infty)$ with respect to a weighted uniform topology. On $\mathcal{D}[0,\infty)$, endowed with the uniform topology, we consider similar subspaces, on which we prove the large deviation principle.

By considering $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function such that

$$\lim_{t\to 0} \frac{\phi(t)}{\sqrt{t|\log t|}} = \lim_{t\to\infty} \frac{\phi(t)}{\sqrt{t\log t}} = \infty, \tag{3.1}$$

the set $\mathcal{D}_\phi$ is defined as

$$\mathcal{D}_\phi := \{\omega \in \mathcal{D}[0,\infty) : \|w\|_\phi = \sup_{t\in\mathbf{R}_+} \frac{|\omega(t)|}{\phi(t)} < \infty\} \tag{3.2}$$

and $\mathcal{M}_1(\mathcal{D}_\phi) := \{Q \in \mathcal{M}_1(\mathcal{D}[0,\infty)) : Q(\mathcal{D}_\phi) = 1\}$.

On $\mathcal{M}_1(\mathcal{D}_\phi)$ define the distance

$$d_\phi(\mu,\nu) := \sup\left\{\left|\int f\,d\mu - \int f\,d\nu\right|, f \in \mathcal{C}(\mathcal{D}_\phi, \mathbb{R}), \|f\|_L \leq 1\right\}, \tag{3.3}$$

with the Lipschitz norm $\|f\|_L := \sup_{\omega \in \mathcal{D}_\phi} |f(\omega)| + \sup_{\omega,\omega' \in \mathcal{D}_\phi, \omega \neq \omega'} \frac{|f(\omega)-f(\omega')|}{\|\omega-\omega'\|_\phi}$. Thus $(\mathcal{M}_1(\mathcal{D}_\phi), d_\phi)$ becomes a metric space.

Lemma 3.1 *Let X_n and Y_n be random variables with values in a metric space (E,d) such that for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d(X_n, Y_n) > \epsilon\} = -\infty. \tag{3.4}$$

Then X_n and Y_n are LDP equivalent (super-exponentially close), which means that if the sequence $\{X_n\}_n$ satisfies LDP with constants $\log n$ and rate function I, then $\{Y_n\}_n$ also satisfies LDP with the same constants and rate function.

Lemma 3.2 *Let X^n and Y^n be two random elements on $\mathcal{D}[0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{\|X^n - Y^n\|_\phi > \varepsilon\} = -\infty. \tag{3.5}$$

Then, the random measures

$$\mu_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{X^k} \quad \textit{and} \quad \nu_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{Y^k}$$

are LDP equivalent.

Proof: We need to verify that

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d_\phi(\mu_n, \nu_n) > \varepsilon\} = -\infty. \tag{3.6}$$

Let $h \in \mathcal{C}(\mathcal{D}_\phi)$ such that $\|h\|_L \leq 1$ and denote $L(n) = \sum_{k=1}^n \frac{1}{k}$. Then,

$$\begin{gathered}
\left|\int h\, d\mu_n - \int h\, d\nu_n\right| \leq \tfrac{1}{\log n} \textstyle\sum_{k=1}^n \tfrac{1}{k} |h(X^k) - h(Y^k)| = \\
\tfrac{1}{\log n} \textstyle\sum_{k=1}^{[n^{\varepsilon/8}]} \tfrac{1}{k} |h(X^k) - h(Y^k)| + \\
\tfrac{1}{\log n} \textstyle\sum_{k=[n^{\varepsilon/8}]+1}^n \tfrac{|h(X^k)-f(Y^k)|}{\|X^k-Y^k\|_\phi} \|X^k - Y^k\|_\phi \leq \\
\tfrac{2}{\log n} L([n^{\varepsilon/8}]) + \tfrac{1}{\log n}(L(n) - L([n^{\varepsilon/8}]) \sup_{[n^{\varepsilon/8}]+1 \leq k \leq n} \|X^k - Y^k\|_\phi \leq \\
\tfrac{2}{\log n}(1 + \log n^{\varepsilon/8}) + \tfrac{1}{\log n}(1 + \log n) \sup_{[n^{\varepsilon/8}]+1 \leq k \leq n} \|X^k - Y^k\|_\phi \leq \\
2(\tfrac{\varepsilon}{8} + \tfrac{\varepsilon}{8}) + 2 \sup_{[n^{\varepsilon/8}]+1 \leq k \leq n} \|X^k - Y^k\|_\phi \leq \tfrac{\varepsilon}{2} + 2\tfrac{\varepsilon}{4} = \varepsilon,
\end{gathered}$$

if $\sup_{[n^{\varepsilon/8}]+1 \leq k \leq n} \|X^k - Y^k\|_\phi \leq \frac{\varepsilon}{2}$ and $n \geq n_0$ for some $n_0 > 0$. Therefore,

$$\begin{aligned}
\mathbb{P}\{d_\phi(\mu_n, \nu_n) > \varepsilon\} &\leq \mathbb{P}\left\{ \sup_{[n^{\varepsilon/8}]+1 \leq k \leq n} \|X^k - Y^k\|_\phi > \frac{\varepsilon}{2} \right\} \\
&\leq \sum_{k=[n^{\varepsilon/8}]+1}^{n} \mathbb{P}\{\|X^k - Y^k\|_\phi > \frac{\varepsilon}{2}\} \\
&\leq \sum_{k=[n^{\varepsilon/8}]+1}^{n} k^{-\frac{8N}{\varepsilon}} k^{-2} \quad \text{(due to (3.5))} \leq cn^{-N},
\end{aligned}$$

for some positive constant c and N sufficiently large. ■

In the following we prove a sequence of LDP equivalences under the following additional conditions:

C1. the continuous function ϕ defined in (3.1) is increasing such that $\phi(0) > \gamma$, for some $\gamma > 0$;

C2. the family of sojourn times $\{\theta_x,\, x \in E\}$ is uniformly integrable (condition $A4$) and

$$\int_n^\infty \bar{F}_x(t)dt \leq e^{-\psi(n)}, \quad \text{such that} \quad \lim_{n\to\infty} \frac{\psi(n)}{\log n} \to \infty.$$

Proposition 3.3 *Assume that $\{X(t),\, t \geq 0\}$ is an ergodic semi-Markov process with ergodic distribution π and $\{X_n,\, n \geq 0\}$ is its embedding Markov chain with stationary distribution ν. Let $f \in L^2(\pi)$ satisfying the conditions of the martingale decomposition theorem (Theorem 2.2).*

Define the empirical measures

$$\mu_n^W = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{W^{(k)}} \quad \textit{and} \quad \tilde{\mu}_n^W = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{\tilde{W}^{(k)}}$$

where

$$W_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} \int_0^{nt} f(X(u)), \quad \tilde{W}_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k.$$

Then the random measures μ_n^W and $\tilde{\mu}_n^W$ are LDP equivalent.

Proof: According to Lemma 3.2, it is enough to check that

$$lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{\|W^{(n)} - \tilde{W}^{(n)}\|_\phi > \varepsilon\} = -\infty. \tag{3.7}$$

For any $T > 0$, we have:

$$\begin{gathered}
\mathbb{P}\{\sup_{0\leq t\leq T} \tfrac{|W_t^{(n)} - \tilde{W}_t^{(n)}|}{\phi(t)} > \varepsilon\} \leq \\
\mathbb{P}\{\sup_{0\leq t\leq T} \tfrac{|\int_0^{nt} f(X(u))\,du - \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k|}{\phi(t)} > \varepsilon\sigma\sqrt{nm}\} \leq \\
\mathbb{P}\{\sup_{0\leq t\leq T} \tfrac{|(nt-\tau_{N(nt)})f(X_{N(nt)})|}{\phi(t)} > \varepsilon\sigma\sqrt{nm}\} \leq \\
\mathbb{P}\{\sup_{0\leq t\leq T} \tfrac{\theta_{N(nt)+1}|f(X_{N(nt)})|}{\phi(t)} > \varepsilon\sigma\sqrt{nm}\} \leq \\
\mathbb{P}\{\sup_{0\leq t\leq T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{nm}\} + \mathbb{P}\{\sup_{0\leq t\leq T} \tfrac{|f(X_{N(nt)})|}{\phi(t)} > \sqrt[4]{nm}\}.
\end{gathered}$$

Since $\left|\frac{N(nt)}{n} - \frac{t}{m}\right|$ converges to zero almost surely, then for any $M > 0$, there exists some $\delta > 0$ such that

$$\mathbb{P}\{N(nt) > \frac{nt}{m} + n\delta\} \leq n^{-M}.$$

For the sake of simplicity in notation, let us assume $\frac{nt}{m} + n\delta$ is its integer part in the following estimates.

$$\mathbb{P}\{\sup_{0\leq t\leq T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{nm}\} \leq \mathbb{P}\{\sup_{0\leq t\leq T} \theta_{N(nt)+1} > \varepsilon\sigma\sqrt[4]{nm},\, N(nt) \leq \tfrac{nt}{m} + n\delta\} + \mathbb{P}\{N(nt) > \tfrac{nt}{m} + n\delta\}$$

On the one hand,

$$\mathbb{P}\{\sup_{0\le t\le T}\theta_{N(nt)+1}>\varepsilon\sigma\sqrt[4]{nm},\, N(nt)\le \tfrac{nt}{m}+n\delta\}\le \mathbb{P}\{\sup_{1\le k\le \frac{nT}{m}+n\delta}\theta_k>\varepsilon\sigma\sqrt[4]{nm}\}\le$$
$$\textstyle\sum_{k=1}^{\frac{nT}{m}+n\delta}\mathbb{P}\{\theta_k>\varepsilon\sigma\sqrt[4]{nm}\}\le \left(\frac{nT}{m}+n\delta\right)\sup_{x\in E}\int_{\varepsilon\sigma\sqrt[4]{nm}}^{\infty}\bar{F}_x(t)\,dt\le$$
$$\left(\tfrac{nT}{m}+n\delta\right)e^{-\psi(\varepsilon\sigma\sqrt[4]{nm})}\le \left(\tfrac{nT}{m}+n\delta\right)e^{-N\log(\varepsilon\sigma\sqrt[4]{nm})}=$$
$$\left(\tfrac{nT}{m}+n\delta\right)(\varepsilon\sigma\sqrt[4]{m})^{-N}n^{-4N},$$

for any $N>0$ and n sufficiently large.

On the other hand, the second probability can be estimated as follows:

$$\mathbb{P}\{\sup_{0\le t\le T}\tfrac{|f(X_{N(nt)})|}{\phi(t)}>\sqrt[4]{nm}\}\le \mathbb{P}\{\sup_{0\le t\le T}|f(X_{N(nt)})|>\gamma\sqrt[4]{nm}\}\le$$
$$\mathbb{P}\{\sup_{0\le t\le T}|f(X_{N(nt)})|>\gamma\sqrt[4]{nm}, N(nt)\le \tfrac{nt}{m}+n\delta\}+\mathbb{P}\{N(nt)>\tfrac{nt}{m}+n\delta\}.$$

$$\mathbb{P}\{\sup_{0\le t\le T}|f(X_{N(nt)})|>\gamma\sqrt[4]{nm}, N(nt)\le \tfrac{nt}{m}+n\delta\}\le$$
$$\textstyle\mathbb{P}\{\sup_{0\le k\le \frac{nT}{m}+n\delta}|f(X_k)|>\gamma\sqrt[4]{nm}\}\le \sum_{k=0}^{\frac{nT}{m}+n\delta}\mathbb{P}\{|f(X_k)|>\gamma\sqrt[4]{nm}\}=$$
$$\textstyle\sum_{k=0}^{\frac{nT}{m}+n\delta}\mathbb{P}\{f^2(X_k)|>\gamma^2\sqrt{nm}\}=\sum_{k=0}^{\frac{nT}{m}+n\delta}\mathbb{E}[f^2(X_k)\mathbb{1}_{\{f^2(X_k)>\gamma^2\sqrt{nm}\}}]=$$
$$\left(\tfrac{nT}{m}+n\delta\right)\int_{\{f^2(x)>\gamma^2\sqrt{nm}\}}f^2(x)\mu P_k(dx)\le$$
$$c\left(\tfrac{nT}{m}+n\delta\right)\int_{\{f^2(x)>\gamma^2\sqrt{nm}\}}f^2(x)\nu(dx)\le c\left(\tfrac{nT}{m}+n\delta\right)e^{-\varphi(\gamma^2\sqrt{nm})}\le$$
$$\text{under the assumption } (iii) \text{ of Theorem 2.2}$$
$$c\left(\tfrac{nT}{m}+n\delta\right)e^{-N'\log(\gamma^2\sqrt{nm})}=c\left(\tfrac{nT}{m}+n\delta\right)(\gamma^2\sqrt{nm})^{-N'},$$

for some $N'>0$ and n large. Combining the above two estimates, and choosing $N'=8N$ and $M=N-1$, we get:

$$\mathbb{P}\left\{\sup_{0\le t\le T}\frac{|W_t^{(n)}-\tilde{W}_t^{(n)}|}{\phi(t)}>\varepsilon\right\}\le C_{m,N,T,\delta}n^{1-N}.$$

Therefore,

$$\lim_{n\to\infty}\frac{\log\left(\mathbb{P}\{\sup_{0\le t\le T}\frac{|W_t^{(n)}-\tilde{W}_t^{(n)}|}{\phi(t)}>\varepsilon\}\right)}{\log n}\le 1-N.$$

■

A new LDP equivalence can be derived as:

Proposition 3.4 *Define the empirical measures*

$$\tilde{\mu}_n^W=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\tilde{W}^{(k)}}\quad \textit{and}\quad \tilde{\mu}_n^S=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\tilde{S}^{(k)}}$$

where

$$\tilde{W}_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k, \quad \tilde{S}_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} \sum_{k=1}^{[nt/m]} f(X_{k-1})\theta_k.$$

Then the random measures $\tilde{\mu}_n^W$ and $\tilde{\mu}_n^S$ are LDP equivalent.

Proof: For any $T > 0$, we have:

$$\mathbb{P}\{\sup_{0\le t\le T} \tfrac{|\tilde{W}_t^{(n)} - \tilde{S}_t^{(n)}|}{\phi(t)} > \varepsilon\} \le$$
$$\mathbb{P}\{\sup_{0\le t\le T} |\textstyle\sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k - \sum_{k=1}^{[nt/m]} f(X_{k-1})\theta_k| > \varepsilon\gamma\sigma\sqrt{nm}.\}$$

As previously done, we use the notations $\frac{nt}{m}$ and $\frac{nt}{m} + n\delta$ in lieu of their integer parts. Thus,

$$\mathbb{P}\{\sup_{0\le t\le T} \tfrac{|\tilde{W}_t^{(n)} - \tilde{S}_t^{(n)}|}{\phi(t)} > \varepsilon\} \le$$
$$\mathbb{P}\{\sup_{0\le t\le T} |\textstyle\sum_{k=1}^{\frac{nt}{m}+n\delta} f(X_{k-1})\theta_k - \sum_{k=1}^{nt/m} f(X_{k-1})\theta_k| > \varepsilon\gamma\sigma\sqrt{nm}\} + \mathbb{P}\{N(nt) > \frac{nt}{m} + n\delta\}.$$

$$\mathbb{P}\{\sup_{0\le t\le T} |\textstyle\sum_{k=1}^{\frac{nt}{m}+n\delta} f(X_{k-1})\theta_k - \sum_{k=1}^{nt/m} f(X_{k-1})\theta_k| > \varepsilon\gamma\sigma\sqrt{nm}\} \le$$
$$\mathbb{P}\{\sup_{0\le t\le T} |\textstyle\sum_{k=\frac{nt}{m}+1}^{\frac{nt}{m}+n\delta} f(X_{k-1})\theta_k| > \varepsilon\gamma\sigma\sqrt{nm}\} \le$$
$$\mathbb{P}\{\sup_{0\le t\le T} \textstyle\sum_{k=\frac{nt}{m}+1}^{\frac{nt}{m}+n\delta} |f(X_{k-1})|\theta_k > \varepsilon\gamma\sigma\sqrt{nm}\} \le$$
$$\mathbb{P}\{\sup_{0\le t\le T} \sup_{k\in[\frac{nt}{m}+1, \frac{nt}{m}+n\delta]} |f(X_{k-1})|\theta_k > \varepsilon\gamma\sigma\tfrac{\sqrt{nm}}{n\delta-1}\} \le$$
$$\mathbb{P}\{\sup_{0\le k\le \frac{nT}{m}+n\delta} |f(X_{k-1})|\theta_k > \varepsilon\gamma\sigma\tfrac{\sqrt{nm}}{n\delta-1}\} \le$$
$$\mathbb{P}\{\sup_{1\le k\le \frac{nT}{m}+n\delta} \theta_k > \varepsilon\sigma\sqrt[4]{nm}\} + \mathbb{P}\{\sup_{0\le k\le \frac{nT}{m}+n\delta} |f(X_k)| > \gamma\tfrac{\sqrt[4]{nm}}{n\delta-1}\}.$$

The first term is estimated in the proof of Proposition 3.3 as

$$\mathbb{P}\left\{\sup_{1\le k\le \frac{nT}{m}+n\delta} \theta_k > \varepsilon\sigma\sqrt[4]{nm}\right\} \le \left(\frac{nT}{m} + n\delta\right)(\varepsilon\sigma\sqrt[4]{m})^{-N} n^{-4N},$$

for N sufficiently large. By the same argument as in the proof of Proposition 3.3, we get:

$$\mathbb{P}\left\{\sup_{0\le k\le \frac{nT}{m}+n\delta} |f(X_k)| > \gamma\tfrac{\sqrt[4]{nm}}{n\delta-1}\right\} \le c\left(\tfrac{nT}{m} + n\delta\right) e^{-\varphi\left(\frac{\gamma^2}{n^2\delta^2}\sqrt{nm}\right)} \le$$
$$c\left(\tfrac{nT}{m} + n\delta\right)\left(\tfrac{\gamma^2\sqrt{nm}}{n^2\delta^2}\right)^{-N'}.$$

Combining all the estimates in the case $N' = 8N$ and $M = 4N - 1$ we get:

$$\mathbb{P}\left\{\sup_{0\le t\le T} \frac{|\tilde{W}_t^{(n)} - \tilde{S}_t^{(n)}|}{\phi(t)} > \varepsilon\right\} \le C_{m,N,T,\delta} n^{-4N}\left(1 + \frac{\gamma}{n\delta}^{1-4N}\right).$$

Therefore,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}\{\sup_{0\le t\le T} \frac{|\tilde{W}_t^{(n)}-\tilde{S}_t^{(n)}|}{\phi(t)} > \varepsilon\}\right)}{\log n} \le 1-4N.$$

■

For $a>0$, define $\vartheta_a : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ by $\vartheta_a \omega(t) = \frac{1}{\sqrt{a}}\omega(at)$, and we say that a measure Q is ϑ-invariant if $Q = Q\circ\vartheta_a^{-1}$.

For $Q \in \mathcal{M}_1(\mathcal{D}_\phi)$, define

$$I(Q) := \begin{cases} \lim_{a\to\infty} \frac{1}{2\log a} h\left(Q\circ|_{[\frac{1}{a},a]}^{-1} \,||\, W\circ|_{[\frac{1}{a},a]}^{-1}\right) & \text{, if } Q \text{ is } \vartheta\text{-invariant,} \\ \infty & \text{, otherwise.} \end{cases} \tag{3.8}$$

where W is the Wiener measure on $\mathcal{D}[0,\infty)$, $|_{[\frac{1}{a},a]}$ is the restriction operator, and for any two probability measures μ and ν, denote by $h(\mu\,||\,\nu)$ the relative entropy of μ with respect to ν,

$$h(\mu\,||\,\nu) = \begin{cases} \int \log(\frac{d\mu}{d\nu}) & \text{, if } \mu \ll \nu, \\ \infty & \text{, otherwise.} \end{cases}$$

Theorem 3.5 *Let $\{X(t),\, t\ge 0\}$ be an ergodic semi-Markov process with invariant measure π, and $f\in L^2(E,\pi)$ satisfying the conditions of Theorem 2.5. Define the additive functional,*

$$W_t^{(n)} := \frac{1}{\sigma\sqrt{nm}} \int_0^{nt} f(X_s)\,ds.$$

The sequence of random measures,

$$\mu_n^W = \frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}\delta_{W^{(k)}},$$

satisfies the large deviation principle with constants $\log n$ *and rate function* $I|_{\mathcal{M}_1(\mathcal{D}_\phi)}$ *defined in (3.8) . That is, for any Borel set $A\subseteq \mathcal{M}_1(\mathcal{D}_\phi)$,*

$$\begin{aligned} -\inf_{A^\circ} I &\le \liminf_{n\to\infty} \frac{1}{\log n}\log \mathbb{P}\{\mu_n\in A\} \\ &\le \limsup_{n\to\infty} \frac{1}{\log n}\log \mathbb{P}\{\mu_n\in A\} \le -\inf_{\bar{A}} I \end{aligned}$$

Proof: Based on the sequence of LDP equivalences established in Propositions 3.3 and 3.4, it suffices to prove the LDP for $\tilde{\mu}_n^S$, the empirical measure associated to the Markov renewal process. Using the martingale decomposition theorem associated with the Markov renewal process (Theorem 2.2), we get $S_t^{(n)} = \sum_{k=1}^{[nt/m]} f(X_{k-1})\theta_k = M_t^{(n)} + R_t^{(n)}$, with the remainder converging to zero in probability at the rate given in the equation (2.4).

Let $\tilde{\mu}_n^S$ and $\tilde{\mu}_n^M$ be the empirical measures associated with $\tilde{S}_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} S_t^{(n)}$ and, respectively, $\tilde{M}_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} M_t^{(n)}$. These sequences are LDP equivalent because

$$\mathbb{P}\{\sup_{0\le t\le T} \frac{|\tilde{S}_t^{(n)} - \tilde{M}_t^{(n)}|}{\phi(t)} > \varepsilon\} \le \mathbb{P}\{\sup_{0\le t\le T} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{nm}\} \le n^{-\gamma},$$

for some $\gamma > 0$ and n sufficiently large, according to (2.4). The martingale defined in (2.5) is a martingale additive functional, therefore using the large deviation result for martingale additive functionals established by M. Heck and F. Maaouia in [11], we get that the sequence of random measures μ_n^W satisfies the LDP with rate function given in (3.8) .

■

4. Conclusion

Motivated by previous results on large deviations for additive functionals of Markov processes ([18, 12]), we investigate the case of semi-Markov processes. The large deviation principle for the sequence of empirical processes defined as logarithmic averaging of some additive functionals of semi-Markov processes is derived from the LDP of martingales additive functionals, through a sequence of transformations that preserve the large deviation principle. This is a level 3 LDP, obtained from the almost sure central limit theorem, and the rate function is given as a specific relative entropy of the Wiener measure on a subspace of $\mathcal{D}[0, \infty)$.

References

1. B. Bercu. On the convergence of moments in the almost sure central limit theorem for martingales with statistical applications. *Stochastic Processes and their Applications*, 111:157–173, 2004.

2. R.N. Bhattacharya. On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. verw. Gebiete.*, 60:185–201, 1982.

3. P. Billingsley. *Convergence of Probability Measures*, Willey, 1999.

4. G.A. Brosamler. An almost everywhere central limit theorem. *Math. Proc. Cambridge Phil. Soc.*, 104:561–574, 1998.

5. F. Chaabane. Version forte du théorème de la limite centrale fonctionnel pour les martingales. *C.R. Acad. Sci. Paris*, 323.

6. F. Chaabane and F Maaouia. Theoremes limites avec poids pour les martingales vectorielles. *Essaim Prob. Stat.*, 4:137–189, 2000.

7. M.D. Donsker and S.R.S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time iv. *Comm. Pure Appl. Math.*, 36:183–212, 1983.

8. I. Gikhman and A. Skorokhod. *The Theory of Stochastic Processes II.* Springer, Berlin, 2004.

9. P.W. Glynn and P.J. Haas. On functional central limit theorems for semi-Markov and related processes. *Communications in Statistics - Theory and Methods*, 33.

10. M.K. Heck. Principles of large deviations for the empirical processes of the Ornstein-Uhlenbeck process. *Journal of Theoretical Probability*, 12:147–179, 1999.

11. M.K. Heck and F. Maaouia. The principle of large deviations for martingale additive functionals of recurrent Markov processes. *Electronic Journal of Probability*, 6:1–26, 2001.

12. A. Korzeniowski and A. Oprisan. Large deviations for additive functionals of Markov processes. *International Journal of Pure and Applied Mathematics*, 53:441–459, 2009.

13. M.T. Lacey and W. Philipp. A note on the almost sure central limit theorem. *Statistics & Probability Letters*, 9:201–205, 1990.

14. M. Lifshits. Almost sure limit theorem for martingales. *Limit Theorems in Probability and Statistics II , J. Bolyai Mathematical Society.*

15. N. Limnios. Estimation of the stationary distribution of semi-Markov processes with Borel state space. *Statistics & Probability Letters*, 76.

16. N. Limnios and G. Oprişan. Invariance principle for an additive functional of a semi-Markov process. *Rev. Roumaine Math. Pures Appl.*, 44.

17. F. Maaouia. Principes d'invariance par moyennisation logarithmique pour les processus de Markov. 29:1859–1902, 2001.

18. A. Oprisan and A Korzeniowski. Large deviations via almost sure clt for functionals of Markov processes. *Stochastic Analysis and Applications*, 30:933–947, 2012.

19. P. Schatte. On the central limit theorem with almost sure convergence. *Prob. Math. Statist.*, 11:237–246, 1991.