Large Deviations via Almost Sure CLT for Functionals of Markov Processes

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We consider additive functionals of continuous time Markov processes and prove that their functional central limit theorems (FCLT) admit almost sure versions based on empirical measures with logarithmic averaging. For the corresponding empirical processes, we prove a large deviation principle (LDP) based on a martingale decomposition, established here for additive functionals of Markov processes.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{B}_E)$ a complete separable metric space, and let $X_t, t \geq 0$ be a homogeneous Markov process defined on Ω with values in E. Let $p(t, x, dy)$ be the transition probability function on E, i.e.,

(i) $p(t, x, dy)$ is a probability measure on $\mathcal{B}_E$ for each pair $(t, x) : \ t > 0, x \in E$

(ii) for each $B \in \mathcal{B}_E$, the function $(t, x) \to p(t, x, B)$ is Borel measurable on $(0, \infty) \times E$

(iii) Chapman-Kolmogorov relation: for all $B \in \mathcal{B}_E$ and for all $t > 0$, $s > 0$

$$p(t + s, x, B) = \int_E p(s, y, B) p(t, x, dy).$$

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Assume that $p(t,x,dy)$ admits an invariant probability measure m on $(E,\mathcal{B}_E)$:

$$m(B) = \int p(t,x,B)\, m(dx)$$

for all $B \in \mathcal{B}_E$.

Define $L^2(E,m) := \{f : E \to \mathbb{R}, \|f\|_2 = \left(\int_E |f|^2\, dm\right)^{1/2} < \infty\}$. Define the transition operators semigroup on the Banach space of all real bounded measurable functions $B(E)$, $P_t : B(E) \to B(E)$ by

$$(P_t f)(x) = \int_E f(y) p(t,x,dy) \quad t > 0,\, x \in E$$

Then $\{P_t,\, t > 0\}$ is a contraction semigroup on $L^2(E,m)$:

$$\|P_t f\|_2^2 = \int (P_t f(x))^2 m(dx) \leq \int (P_t f^2)(x) m(dx) \leq \int f^2(x)\, m(dx) = \|f\|_2^2.$$

Let L be the infinitesimal generator of $(X_t,\, t \geq 0)$ with domain $\mathcal{D}(L) \subset L^2(m)$

$$Lf := \lim_{t \to 0} \frac{P_t f - f}{t}$$

with the limit being in sup norm. Since $\|\cdot\|_2$ is weaker than sup norm, we consider $\hat{L}$ the extension of L on $L^2(E,m)$.

Theorem 1.1 *(FCLT, Bhathacharya [2]) Let $\{X_t,\, t \geq 0\}$ be a progressively measurable stationary ergodic Markov process having transition probability function $p(t,x,dy)$ and invariant initial distribution m. If $f \in \mathcal{R}(\hat{L})$, the range of its infinitesimal generator, then*

$$S_t^{(n)} = \frac{1}{\sqrt{n}} \int_0^{nt} f(X_s)\, ds \Rightarrow W$$

where W is the Wiener measure with zero drift and variance parameter

$$\sigma^2 = -2 < f, g >= -2 \int f(x) g(x)\, m(dx),$$

and g is an element of domain $\mathcal{D}(\hat{L})$.

If the process is not in equilibrium (i.e. the initial distribution $\neq m$), then the FCLT still holds under the additional condition of the convergence in norm of the transition probability function

$$\|p(t,x,dy) - m(dy)\| \to 0$$

as $t \to 0$, and for all $x \in E$ as was established in [2].

The FCLT have been studied by many authors, notably by Kipnis and Varadhan [11] who proved the FCLT for additive functionals of stationary reversible ergodic Markov chains. A recent result that extends Theorem 1.1 is due to Ferre et al. [7], where the FCLT holds under the following assumptions:

- Markov process $(X_t)_{t\geq 0}$ is stationary with initial distribution m
- Markov semigroup $(P_t)_{t\geq 0}$ has spectral gap on $L^2(m)$

$$lim_{t\to\infty}\|P_t - \Pi\|_2 = 0$$

  where Π denotes the projection on $L^2(m)$, $\Pi f = m(f)\mathbb{1}_E$
- $S_t := \int_0^t f(X_s)\,ds$ satisfies the moment condition

$$\sup_{t\in(0,1]} \mathbb{E}_m(|S_t|^2) < \infty.$$

If the process is uniformly ergodic, i.e., $lim_{t\to\infty}\|P_t - \Pi\|_\infty = 0$ then the FCLT also follows.

In Cattiaux et al. [4] a FCLT for a Markov diffusion process X_t on $\mathbb{R}^d$ is established. The authors provide an alternative approach which is not based on the resolvent as in Kipnis and Varadhan [11], and instead they use a martingale approximation via Poisson equation in $L^2(m)$. A criterion for the existence of the solution of the Poisson equation is given:

Theorem 1.2 ([4]) *Let $f \in L^2(m)$ with $\int f\,d\mu = 0$. If $\int_0^\infty s\|P_s f\|_{L^2(m)}\,ds < \infty$ then $f \in \mathcal{D}(L^{-1})$ and $g(\cdot) = -\int_0^\infty P_s f(\cdot)\,ds$ solves the Poisson equation $Lg = f$.*

2. Martingale decomposition

In [13] we proved a martingale decomposition for a class of additive functionals of Markov chains. In what follows we apply analogous method in continuous time setting.

Assume $f \in L^2(m)$ such that $\int f\,dm = 0$ and define

$$S_t = \int_0^t f(X_s)\,ds \tag{2.1}$$

Then $S_t \in L^2(m) \subset L^1(m)$.

We say that X is a solution of the martingale problem for L with domain $\mathcal{D}(L)$ if and only if there is a filtration $\mathcal{F}_t, t \geq 0$ such that for any $g \in \mathcal{D}(L)$

$$Z_t := g(X_t) - g(X_0) - \int_0^t (Lg)(X_s)\,ds$$

is a local L^2-martingale with respect to the filtration $\mathcal{F}_t$. If Lf is bounded then Z_t is a zero mean martingale.

Assume that there exists g solving the Poisson equation $Lg = f$ in $L^2(m)$, and without loss of generality assume that $\int g\,dm = 0$. Then S_t has the following decomposition

$$S_t = M_t + R_t$$

where M_t is a local L^2-martingale, $R_t = g(X_t) - g(X_0)$, $M_t = -Z_t$ and $M_0 = R_0 = 0$.

In this section we want to prove a martingale decomposition theorem for the additive functionals of Markov processes defined as

$$S_t^{(n)} := \frac{1}{\sigma\sqrt{n}} \int_0^{nt} f(X_s)\, ds. \tag{2.2}$$

Definition 2.1 *$f : \mathbb{R} \to \mathbb{R}$ is in the H_α class if there exists $B > 1$ such that for large $\|x\|$*

$$\frac{1}{B} h(\|x\|) < |f(x)| < B h(\|x\|) \tag{2.3}$$

where $h : [0, \infty) \to [0, \infty)$ is continuous such that

(i) $h(b) < Bb^\alpha$, for some $1 \le \alpha < \infty$

(ii) $\frac{h(b)}{b}$ is nondecreasing

for large b.

Remark 2.2 *$h(\cdot)$ is strictly increasing and $\frac{h(b)}{B} \ge h(\frac{b}{B})$.*

Lemma 2.3 *Let $\{X_t,\, t \ge 0\}$ be a Markov process such that for any $0 < T < \infty$ and large b with $\|X_0\| < b$,*

$$\mathbb{P}\{\|X_T\| > b \mid \sup_{0 \le t \le T} \|X_t\| > b\} \ge \frac{1}{A},\ 1 < A < \infty. \tag{2.4}$$

Then for any $f \in H_\alpha$,

$$\mathbb{P}\{\sup_{t \in [0,T]} |f(X_t)| > b\} \le A\mathbb{P}\left\{|f(X_T)| > h\left(\frac{b}{B^3}\right)\right\}. \tag{2.5}$$

Proof: Given $\frac{1}{A} \le \mathbb{P}\left\{\|X_T\| > \frac{b}{B} \mid \sup_{0\le t\le T} \|X_t\| > \frac{b}{B}\right\} \le \frac{\mathbb{P}\{\|X_T\| > \frac{b}{B}\}}{\mathbb{P}\{\sup_{0\le t\le T}\|X_t\| > \frac{b}{B}\}}$ we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{0\le t\le T} h(\|X_t\|) > h\left(\frac{b}{B}\right)\right\} &\le \mathbb{P}\left\{\sup_{0\le t\le T}\|X_t\| > \frac{b}{B}\right\} \le A\mathbb{P}\left\{\|X_T\| > \frac{b}{B}\right\} \\
&\le A\mathbb{P}\left\{h(\|X_T\|) > h\left(\frac{b}{B}\right)\right\}.
\end{aligned}$$

By Definition 2.1 and Remark 2.2 we get

$$\begin{aligned}
&\mathbb{P}\{\sup_{0\le t\le T} |f(X_t| > Bb\} \le \mathbb{P}\{\sup_{0\le t\le T} |f(X_t)| > h(b)\} \le \mathbb{P}\left\{\sup_{0\le t\le T} h(\|X_t\|) > \frac{h(b)}{B}\right\} \le \\
&\mathbb{P}\left\{\sup_{0\le t\le T} h(\|X_t\|) > h\left(\frac{b}{B}\right)\right\} \le A\mathbb{P}\left\{h(\|X_T\|) > h\left(\frac{b}{B}\right)\right\} \le A\mathbb{P}\left\{|f(X_T)| > h\left(\frac{b}{B^2}\right)\right\}
\end{aligned}$$

whence setting $b \to \frac{b}{B}$ gives (2.5). ■

Theorem 2.4 *Let $\{X_t,\, t \geq 0\}$ be an ergodic E-valued Markov process with property* (2.4)*, initial distribution μ and unique invariant measure m. Let $f \in L^2(m) = \{f : E \to \mathbb{R},\, \int_E f^2\, dm < \infty\}$ such that*

(i) $\int_E f\, dm = 0$ *and* $f \in H_\alpha$,

(ii) $||P_t f||_{L^2(m)} \leq \rho^t ||f||_{L^2(m)}$ *for some* $0 < \rho < 1$,

(iii) $d\mu P_t \leq D dm < \infty$ *for some* $0 < D < \infty$,

(iv) $|P_t f(x)| \leq Ca$ *whenever* $|f(x)| \leq a$ *for some* $1 < C < \infty$, $x \in E$ *for large* a, $\int_{\{x: f^2(x) > a\}} f^2(x)\, m dx \leq e^{-\varphi(a)}$ *for large* a, *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\lim_{a\to\infty} \frac{\varphi(a)}{\log a} = \infty$.

Then

$$\int_0^{nt} f(X_s)\, ds = M_t^{(n)} + R_t^{(n)}$$

where $\{M_t^{(n)},\, t \geq 0\}$ *is a local* L^2*- martingale relative to* $(\Omega, \mathcal{F}_t, \mathbb{P})$, $\mathcal{F}_t = \sigma(X_s : s \leq t)$ *and*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left\{ \sup_{t\in[0,1]} \frac{|R_t^{(n)}|}{\sigma\sqrt{n}} > \varepsilon \right\} = -\infty \tag{2.6}$$

Proof: Condition (ii) implies that $\int_0^\infty s\|P_s f\|_{L^2(m)}\, ds < \infty$ so the conditions of Theorem (1.2) are fulfilled. It follows that there exists a solution of the Poisson equation and the martingale decomposition yields

$$\int_0^{nt} f(X_s)\, ds = M_t^{(n)} + R_t^{(n)}$$

where $R_t^{(n)} = g(X_{nt}) - g(X_0)$. Furthermore,

$$S_t^{(n)} := \frac{1}{\sigma\sqrt{n}} \int_0^{nt} f(X_s)\, ds = \tilde{M}_t^{(n)} + \frac{1}{\sigma\sqrt{n}} R_t^{(n)}$$

with $\tilde{M}_t^{(n)} = \frac{M_t^{(n)}}{\sigma\sqrt{n}}$ and $R_t^{(n)} = g(X_{nt}) - g(X_0) = -\int_0^\infty ((P_s f)(X_{(nt)}) - (P_s f)(X_0))\, ds$. To prove (2.6) we proceed as follows.

$$\mathbb{P}\left\{ \sup_{t\in[0,1]} \frac{|R_t^{(n)}|}{\sigma\sqrt{n}} > C\varepsilon \right\} = \mathbb{P}\left\{ \sup_{1\leq k\leq n} \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} \frac{|R_t^{(n)}|}{\sigma\sqrt{n}} > C\varepsilon \right\} \leq$$

$$\sum_{k=1}^{n} \mathbb{P}\left\{ \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} |g(X_{nt}) - g(X_0)| > C\varepsilon\sigma\sqrt{n} \right\} \leq$$

$$\sum_{k=1}^{n} \mathbb{P}\left\{ \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} \left| \int_0^{n^{1/4}} [P_s f(X_{nt}) - P_s f(X_0)]\, ds \right| > C\varepsilon \frac{\sigma\sqrt{n}}{2} \right\} +$$

$$\sum_{k=1}^{n} \mathbb{P}\left\{ \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} \left| \int_{n^{1/4}}^{\infty} [P_s f(X_{nt}) - P_s f(X_0)]\, ds \right| > C\varepsilon \frac{\sigma\sqrt{n}}{2} \right\}$$

Let $b = \frac{\varepsilon\sigma n^{1/4}}{4}$, then for each term of the first sum we have

$$\mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_0^{n^{1/4}} |P_s f(X_{nt}) - P_s f(X_0)|\, ds > 2Cbn^{1/4}\right\} \leq$$
$$\mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_0^{n^{1/4}} |P_s f(X_{nt})|\, ds > Cbn^{1/4}\right\} + \mathbb{P}\left\{\int_0^{n^{1/4}} |P_s f(X_0)|\, ds > Cbn^{1/4}\right\} \leq$$
$$\mathbb{P}\left\{\sup_{u\in[k-1,k]}\int_0^{n^{1/4}} |P_s f(X_u)|\, ds > Cbn^{1/4}\right\} + \mathbb{P}\left\{\int_0^{n^{1/4}} |P_s f(X_0)|\, ds > Cbn^{1/4}\right\}.$$

By (iv), if $|f(X_u)| \leq b$ implies $|P_s f(X_u)| \leq Cb$ and integrating over $[0, n^{1/4}]$, we get $\int_0^{n^{1/4}} |P_s f(X_u)|\, ds \leq Cbn^{1/4}$ so

$$\mathbb{P}\left\{\sup_{u\in[k-1,k]}\int_0^{n^{1/4}} |P_s f(X_u)|\, ds > Cbn^{1/4}\right\} \leq \mathbb{P}\{\sup_{u\in[k-1,k]} |f(X_u)| > b\} \tag{2.7}$$

Using Lemma 2.3 and setting $r = r(b) = h(\frac{b}{B^3}) > 1$ for sufficiently large b, we get

$$\mathbb{P}\{\sup_{u\in[k-1,k]} |f(X_u)| > b\} \leq A\mathbb{P}\left\{|f(X_k)| > h\left(\frac{b}{B^3}\right)\right\} \leq \frac{A}{r}\mathbb{E}\{|f(X_k)|\mathbb{1}_{\{|f(X_k)>r\}}\} =$$
$$\frac{A}{r}\int_{|f(x)|^2>r^2} |f(x)\mu P_k(dx) \leq A\cdot De^{-\varphi(r^2)}$$

Similarly,

$$\mathbb{P}\left\{\int_0^{n^{1/4}} |P_s f(X_0)|\, ds > Cbn^{1/4}\right\} \leq \mathbb{P}\{|f(X_0)| > b\} \leq D\exp(-\varphi(b^2)).$$

Thus,

$$\sum_{k=1}^{n}\mathbb{P}\left\{\left|\int_0^{n^{1/4}} [P_s f(X_{nt}) - P_s f(X_0)]\, ds\right| > C\varepsilon\frac{\sigma\sqrt{n}}{2}\right\} \leq n[C_1 e^{-\varphi(r^2)} + C_2 e^{-\varphi(b^2)}]. \tag{2.8}$$

On the other hand, for the second term we have

$$\mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_{n^{1/4}}^{\infty} |P_s f(X_{nt}) - P_s f(X_0)|\, ds > 2Cbn^{1/4}\right\} \leq \mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_{n^{1/4}}^{\infty} |P_s f(X_{nt})|\, ds > Cbn^{1/4}\right\} +$$
$$\mathbb{P}\left\{\int_{n^{1/4}}^{\infty} |P_s f(X_0)|\, ds > Cbn^{1/4}\right\}$$

Using properties (ii) and (iii) we further get

$$\mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_{n^{1/4}}^{\infty}|P_s f(X_{nt})|\,ds > Cbn^{1/4}\right\} \le \frac{1}{Cbn^{1/4}}\mathbb{E}\left(\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\int_{n^{1/4}}^{\infty}|P_s f(X_{nt})|\,ds\right) \le$$
$$\frac{1}{Cbn^{1/4}}\int_{n^{1/4}}^{\infty}\mathbb{E}(\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}|P_s f(X_{nt})|)\,ds \le \frac{1}{Cbn^{1/4}}\int_{n^{1/4}}^{\infty}\int(\sup_{u\in[k-1,k]}|P_s f(x)|\mu P^u(dx))\,ds \le$$
$$\frac{1}{Cbn^{1/4}}\int_{n^{1/4}}^{\infty}D\|P_s f\|_1 ds \le \frac{1}{Cbn^{1/4}}\int_{n^{1/4}}^{\infty}D\|P_s f\|_2\,ds \le \frac{1}{Cbn^{1/4}}\int_{n^{1/4}}^{\infty}D\rho^s\|f\|_2 ds \le \frac{D\|f\|_2}{Cbn^{1/4}}\frac{\rho^{n^{1/4}}}{\log\frac{1}{\rho}}.$$

Similarly,

$$\mathbb{P}\left\{\int_{n^{1/4}}^{\infty}|P_s f(X_0)|\,ds > \frac{C}{2}\varepsilon\sigma\sqrt{n}\right\} \le \frac{D\|f\|_2}{Cbn^{1/4}}\frac{\rho^{n^{1/4}}}{\log\frac{1}{\rho}}.$$

It follows that there exists positive constant C_3 such that

$$\sum_{k=1}^{n}\mathbb{P}\left\{\sup_{t\in[\frac{k-1}{n},\frac{k}{n}]}\left|\int_{n^{1/4}}^{\infty}[P_s f(X_{nt}) - P_s f(X_0)]\,ds\right| > C\varepsilon\frac{\sigma\sqrt{n}}{2}\right\} \le C_3\frac{n\rho^{n^{1/4}}}{bn^{1/4}} \le C_4\sqrt{n}e^{-n^{1/4}\gamma} \tag{2.9}$$

where $\gamma = -\log\rho > 0$. Combining equations (2.8) and (2.9) we get

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\frac{|R_t^{(n)}|}{\sigma\sqrt{n}} > C\varepsilon\right\} \le n[C_1 e^{-\varphi(r^2)} + C_2 e^{-\varphi(b^2)}] + C_4\sqrt{n}e^{-n^{1/4}\gamma}.$$

This shows that (2.6) holds because $\lim_{n\to\infty}\frac{\varphi(r^2)}{\log n} = \infty$ and $\lim_{n\to\infty}\frac{\varphi(b^2)}{\log n} = \infty$. Indeed, $\frac{\varphi(r^2)}{\log n} = \frac{\varphi(r^2)}{\log r^2}\cdot\frac{logr^2}{\log n}$ with the first factor going to infinity according to assumption (iv), while $\frac{\log r^2}{\log n} \ge \frac{1}{4} > 0$ for large n due to $h^2(\frac{b}{B^3}) \ge h^2(1)\frac{b^2}{B^6} = \frac{\varepsilon^2\sigma^2\sqrt{n}}{16B^6}h^2(1) = const\cdot\sqrt{n}$.
The same argument implies $\lim_{n\to\infty}\frac{\varphi(b^2)}{\log n} = \infty$. ■

Corollary 2.5 *Some processes that satisfy the conclusions of Theorem 2.4:*

- (a) *finite state irreducible continuous-time Markov chains* (X_t)
- (b) *continuous-time Markov processes that are symmetric on* $L^2(m)$ *(i.e. for which* m *is a reversible measure)*
- (c) *uniformly ergodic Markov processes with bounded functions* f*.*

Proof:

(a). Since there exists a unique invariant measure $m = (m_1, ..., m_N)^T$, the assumption (i), $\int_E f\,dm = 0$ reads $\sum_{i=1}^N f_i m_i = m^T f = 0$, where $f = (f_1, ..., f_N)^T$. By By Fredholm Alternative for the generator $L \equiv Q = (q_{ij})$, $1 \le 1, j \le N$, the equation $Qg = f$ has a solution if and only if $f \perp h$ where

$Q^T h = 0$. This holds because $mQ = 0$ or $Q^T m^T = 0$. Regarding $f \in H_\alpha$, it is always satisfied on a finite state space. There is no need to verify (ii) since that was assumed to guarantee a solution g to $Lg = f$. Moreover (iii) and (iv) hold because f is bounded and Lemma 2.3 is not needed.

(b). For the sake of the discussion, we consider a class of continuous time Markov processes in $(\mathbb{R}^d, |\cdot|)$ such that $P_t f = e^{tA} f$ for smooth functions f, where the infinitesimal generator A has a discrete spectrum $\{-\lambda_n, n \in \mathbb{N}\}$ with $\lambda_0 = 0 < \lambda_1 < \lambda_2 \cdots$ and its corresponding orthonormal in $L^2(m)$ set of eigenfunctions $\{e_n, n \in \mathbb{N}\}$ with $e_0 = 1$. Then for any f in the orthogonal complement $1^\perp$, the condition (i) is satisfied and $P_t e_n = e^{-\lambda_n t} e_n$, $n \geq 1$. Consequently, for $f \in 1^\perp$, $f = \sum_{n=1}^\infty \alpha_n e_n$, $P_t f(x) = \sum_{n=1}^\infty \alpha_n e^{-\lambda_n t} e_n$, whence $\|P_t f(x)\|_{L^2(m)} \leq \rho^t \|f\|_{L^2(m)}$ with $0 < \rho = e^{-\lambda_1} < 1$ and this gives (ii). Furthermore, the invariant measure m often has a density with respect to Lebesgue measure and condition $\frac{d\mu P_t}{dm} \leq D < \infty$ is satisfied. Finally, given m, the tail condition $\int_{\{f^2 > n\}} f^2 \, dm \leq \exp(-\varphi(n))$ is readily verifiable for a large class of $f \in 1^\perp$ in addition to which f is chosen to satisfy (iv).

(c) If the Markov process is uniformly ergodic then the potential operator $R = \int_0^\infty (P_t - \Pi) \, dt$ is a bounded operator and $g = Rf$ is the unique solution of the Poisson equation $Lg = f$ (see [12]), with projector operator on $L^2(m)$ defined by $\Pi f(x) = \int m(dy) f(y) \mathbb{1}(x)$. Since R is bounded the conclusions of the martingale decomposition theorem follows directly.

■

Example.

Consider a 1-dimensional Ornstein-Uhlenbeck process X_t satisfying the equation $dX_t = -\frac{1}{2} X_t dt + dB_t$, $X_0 = x$, where B_t is the standard Brownian motion, with infinitesimal generator $A = \frac{1}{2} \frac{d^2}{dx^2} - \frac{1}{2} x \frac{d}{dx}$ and the invariant measure $dm = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Then $P_t f(x) := \int f(y) p(t, x, y) dy$, where

$$p(t, x, y) = \frac{1}{\sqrt{2\pi(1 - e^{-t})}} e^{-\frac{(xe^{-\frac{t}{2}} - y)^2}{2(1 - e^{-t})}}$$

is a m-symmetric hypercontractive Hermite semigroup.

Here, for Hermite polynomials $H_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{n}}$, we have $H_0 = 1, H_1 = x, H_2 = x^2 - 1, H_3 = x^3 - 3x, \ldots$, $e_n = \frac{H_n}{\sqrt{n!}}$ form an orthonormal basis for $L^2(m)$ and $AH_n = -\frac{n}{2} H_n$ gives the corresponding eigenvalues $\lambda_n = -\frac{n}{2}$, $n \in \mathbb{N}$. Since

$$d\mu P_t = \frac{1}{\sqrt{2\pi(1 - e^{-t})}} e^{-\frac{(xe^{-\frac{t}{2}} - y)^2}{2(1 - e^{-t})}} dy$$

then, for example, taking $\mu = \delta_x = \delta_0$ we have

$$\frac{d\mu P_t}{dm} = \frac{1}{\sqrt{1 - e^{-t}}} e^{-\frac{y^2}{2} \left(\frac{e^{-t}}{1 - e^{-t}} \right)} \leq \sqrt{\frac{e}{e - 1}} \equiv D.$$

It is easy to see that for polynomial domination, $|f(x)| \leq B|x|^l$, $l \in \mathbb{N}$, and for large n

$$\int_{\{f^2 > n\}} f^2 \, dm \leq C n^{2l-1} e^{-\frac{n^2}{2}} \leq e^{-\frac{n^2}{4}}$$

whence one may choose $\varphi(x) = \frac{x^2}{4}$ to satisfy the tail estimate.
For example, taking $f(x) = x$ one gets

$$|P_t f(x)|^2 = \left(\int |f(y)p(t,x,y)\,dy| \right)^2 \le 1 - e^{-t} + x^2 \le 4max(1, x^2)$$

and condition (iv) holds true with $C = 2$.
To see why (2.4) holds for the Ornstein-Uhlenbeck starting at $X_0 = 0$ observe that $X_t = \int_0^t e^{-\frac{1}{2}(t-s)}\,dW_s$, which after changing a clock is a Brownian motion, thanks to reflection principle satisfies $\mathbb{P}\{|X_T| > b \mid \sup_{0\le t\le T} |X_t| > b\} > \frac{1}{2}$ or $A = 2$.

3. Almost sure functional central limit theorem

Let us consider a sequence of random variables $X_n, n \ge 0$ and their corresponding empirical distributions $Q_n = \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k}\,\delta_{X_k}$ where $L(n) = \sum_{k=1}^n \frac{1}{k}$. Statements regarding the convergence of the empirical distributions to a limit distribution with probability one are referred to as almost sure limit theorems. One may replace $L(n)$ by $\log n$. Whenever the limit distribution is Gaussian, the convergence is referred to as an almost sure central limit theorem (ASCLT). Almost sure central limit theorems for sequences of independent and identically distributed random variables were established by Brosamler [3], Shatte [19] and Lacey and Phillip [14].

We state here the functional almost sure central limit theorem (FASCLT) for Brownian motion.

Theorem 3.1 ([*3*]) *Let B be a one-dimensional Brownian motion starting at 0 on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define the $\mathcal{C}[0,1]$-valued random variables $B^{(s)}$, $s > 0$*

$$B_u^{(s)} = \frac{1}{\sqrt{s}} B_{us}, \quad u \in [0,1].$$

By averaging the random point measure $\delta_{B^{(s)}(\omega)}$ in $\mathcal{C}[0,1]$ on a logarithmic time scale, we get the random measures

$$\mu_t(\omega) = \frac{1}{\log t} \int_1^t \frac{ds}{s} \delta_{B^{(s)}(\omega)}, \quad t > 1$$

and their discrete analogs

$$\nu_n(\omega) = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{B^{(k)}(\omega)}, \quad n \ge 2.$$

Then

(a) $\lim_{t\to\infty} \mu_t = W \quad \mathbb{P} - a.e.$

(b) $\lim_{n\to\infty} \nu_n = W \quad \mathbb{P} - a.e.$

where W is the Wiener measure on $\mathcal{C}[0,1]$.

Almost sure central limit theorems versions for martingales were established by Maaouia [16], Chaabane [5], Chaabane and Maaouia [6], Lifshits [15] and Bercu [1]. In [13] we proved an almost sure CLT result for a class of additive functionals for discrete-time Markov chains using a martingale approximation. Here we apply the martingale decomposition for continuous-time Markov processes, Theorem 2.4, in order to establish an almost sure version of the FCLT given in Theorem 1.1. Due to Corollary 2.5, Theorem 3.2 is also an almost sure version of the functional CLT given in [7].

Theorem 3.2 (*FASCLT*) *Let $\{X_t,\, t \geq 0\}$ be an ergodic Markov process with initial distribution μ and unique invariant measure m. Let $f \in L^2(m)$ satisfies the conditions of the martingale decomposition Theorem 2.4, and let $S_t^{(u)} = \frac{1}{\sqrt{u}} \int_0^{ut} f(X_s)\, ds$. The corresponding empirical measures are defined by*

$$W_t(\omega) = \frac{1}{\log t} \int_1^t \frac{du}{u} \delta_{S^{(u)}(\omega)}$$

and

$$W_n(\omega) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{S^{(k)}(\omega)}, \quad n \geq 2.$$

Then

(*a*) $\lim_{t\to\infty} W_t = W, \quad \mathbb{P} - a.e.$

(*b*) $\lim_{n\to\infty} W_n = W, \quad \mathbb{P} - a.e.$

where W is the Wiener measure on $\mathcal{C}[0,\infty)$.

Proof: From Theorem 2.4,

$$S_t^{(n)} = \frac{1}{\sigma\sqrt{n}} M_t^{(n)} + \frac{1}{\sigma\sqrt{n}} R_t^{(n)}$$

where $M_n(t)$ is a local L^2-martingale. Let's denote $\tilde{M}_t^{(n)} := \frac{1}{\sigma\sqrt{n}} M_t^{(n)}$ and let $W_n^{\tilde{M}}$ be the empirical measure associated with the local L^2-martingale $\tilde{M}_t^{(n)}$. If we assume that on $\mathcal{C}[0,\infty)$

$$W_n^{\tilde{M}} \Rightarrow W, \quad \mathbb{P} - a.e. \tag{3.1}$$

then to prove that $W_n(\cdot)$ converges weakly to the Wiener measure, we need for any bounded continuous function $h : \mathcal{C}[0,\infty) \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \delta_{h(S^{(k)})} = \lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \delta_{h(\tilde{M}^{(k)})} \quad \mathbb{P} - a.e.$$

This is equivalent to showing

$$\lim_{n\to\infty} ||S^{(n)} - \tilde{M}^{(n)}||_{\mathcal{C}[0,\infty)} = 0 \quad \mathbb{P} - a.e.$$

Let $A_n = \{\omega : sup_{t\in[0,1]} \frac{|R_t^{(n)}(\omega)|}{\sigma\sqrt{n}} \geq \varepsilon\}$. From (2.6), we have $\mathbb{P}(A_n) \leq n^{-a_n}$ where a_n is a sequence converging to ∞. Since $\sum_{n=1}^{\infty} n^{-a_n} < \infty$ using Borrel-Cantelli lemma we get that A_n converges to 0 $\mathbb{P}$-a.s.. This gives

$$\lim_{n\to\infty} \|S^{(n)} - \tilde{M}^{(n)}\|_{\mathcal{C}[0,1]} = 0 \quad \mathbb{P} - a.s.$$

and implies that W_n converges weakly to the Wiener measure on $\mathcal{C}[0,1]$, $\mathbb{P} - a.e.$. By replacing 1 with N in the above, one can show that the result holds for $\mathcal{C}[0,N]$, for every $N > 0$. The extension to $\mathcal{C}[0,\infty)$ is standard: a sequence ν_n converges weakly to ν on $\mathcal{C}[0,\infty)$ if for every $N > 0$, $\pi_N\nu_n$ converges weakly to $\pi_N\nu$ on $\mathcal{C}[0,N]$, where π_N of a measure on $\mathcal{C}[0,\infty)$ denotes the measure it induces on $\mathcal{C}[0,N]$.

To complete the proof it remains to verify (3.1). Let us define the stopping time

$$T_t := \inf\{s : \langle M^{(n)}\rangle_s > t\}$$

where $\langle M^{(n)}\rangle_t$ is the quadratic variation of the local martingale $M_t^{(n)}$. According to Dambis-Dubins-Schwarz theorem (see Theorem 1.6, [18]), $B_t = M_{T_t}^{(n)}$ is a $(\mathcal{F}_{T_t})$-Brownian motion and $M_t^{(n)} = B_{\langle M^{(n)}\rangle_t}$. We have

$$\tilde{M}_t^{(n)} = \frac{1}{\sigma\sqrt{n}} M_t^{(n)} = \frac{1}{\sigma\sqrt{n}} B_{\langle M^{(n)}\rangle_t}$$

whence (3.1) holds by Theorem 3.1 and proves 3.2b while the proof of 3.2a is similar. ■

4. Large deviation principle from an almost sure central limit theorem

In this section we establish a large deviation principle (LDP) for the sequence of empirical measures

$$W_n(\omega) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{S^{(k)}(\omega)}$$

that satisfy the functional almost everywhere central limit theorem 3.2. In [13] we proved a LDP result for a class of empirical measures associated with additive functionals of the form

$$S_n(f) = \sum_{k=0}^{n-1} f(X_k)$$

where (X_k) is a Markov chain with an invariant measure m and $f \in L^2(m)$. A Donsker-Varadhan type rate function has been derived and in what follows we will show a similar result.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that

$$\lim_{t\to 0} \frac{\phi(t)}{\sqrt{t|\log t|}} = \lim_{t\to\infty} \frac{\phi(t)}{\sqrt{t\log t}} = \infty \tag{4.1}$$

and the set $\mathcal{C}_\phi$ defined as

$$\mathcal{C}_\phi := \{\omega \in \mathcal{C}[0,\infty) : \sup_{t\in\mathbf{R}_+} \frac{|\omega(t)|}{\phi(t)} < \infty\} \tag{4.2}$$

$\mathcal{M}_1(\mathcal{C}_\phi) := \{Q \in \mathcal{M}_1(\mathcal{C}[0,\infty)) : Q(\mathcal{C}_\phi) = 1\}$

Lemma 4.1 *Let Y_n and Z_n be random variables with values in a metric space (E,d) such that for all $\epsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d(Y_n, Z_n) > \epsilon\} = -\infty \tag{4.3}$$

Then Y_n and Z_n are equivalent with respect to LDP (or exponentially equivalent), which means that if (Y_n) satisfies LDP with constants $(\log n)$ and rate function I, then (Z_n) also satisfies LDP with the same constants and rate function.

To use this lemma for sequences of random variables in $\mathcal{M}_1(\mathcal{C}_\phi)$ it is necessary to define a metric, as done in [8].

Let $(\mathcal{C}_\phi, |\cdot|_\phi)$ be a metric space, with $|\omega|_\phi = \sup_{t\in\mathbb{R}_+} \frac{|\omega(t)|}{\phi(t)}$. On $\mathcal{M}_1(\mathcal{C}_\phi)$ define

$$d_\phi(\mu,\nu) := \sup\left\{ \left| \int f\, d\mu - \int f\, d\nu \right|, f \in \mathcal{C}(\mathcal{C}_\psi, \mathbb{R}), \|f\|_L \le \frac{1}{2} \right\} \tag{4.4}$$

where $\|f\|_L := \sup_{\omega\in\mathcal{C}_\phi} |f(\omega)| + \sup_{\omega,\omega'\in\mathcal{C}_\phi, \omega\neq\omega'} \frac{|f(\omega)-f(\omega')|}{|\omega-\omega'|_\phi}$. Then $(\mathcal{M}_1(\mathcal{C}_\phi), d_\phi)$ becomes a metric space and the following properties hold:

(a) $d_\phi \le 1$

(b) $d_\phi(\alpha\mu + (1-\alpha)\nu, \alpha\tilde{\mu} + (1-\alpha)\tilde{\nu}) \le \alpha d_\phi(\mu,\tilde{\mu}) + (1-\alpha) d_\phi(\nu,\tilde{\nu})$, for $\alpha \in [0,1]$ and $\mu, \tilde{\mu}, \nu, \tilde{\nu} \in \mathcal{M}_1(\mathcal{C}_\phi)$

(c) $d_\phi(\delta_\omega, \delta_{\omega'}) \le |\omega - \omega'|_\psi$ for $\omega, \omega' \in \mathcal{C}_\phi$.

Lemma 4.2 *If*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{|S^{(n)} - \tilde{M}^{(n)}|_\phi > \varepsilon\} = -\infty \tag{4.5}$$

then $W_n = \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} \delta_{S^{(k)}}$ and $W_n^{\tilde{M}} = \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} \delta_{\tilde{M}^{(k)}}$ are LDP equivalent.

Proof: For $f \in \mathcal{C}(\mathcal{C}_\phi, \mathbb{R})$, $\|f\|_L \le \frac{1}{2}$ we have

$$\left| \int f\, dW_n - \int f\, dW_n^{\tilde{M}} \right| \le \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} |f(S^{(k)}) - f(\tilde{M}^{(k)})| =$$

$$\frac{1}{L(n)} \sum_{k=1}^{[n^{\varepsilon/2}]} \frac{1}{k} |f(S^{(k)}) - f(\tilde{M}^{(k)})| + \frac{1}{L(n)} \sum_{k=[n^{\varepsilon/2}]+1}^{n} \frac{1}{k} \frac{|f(S^{(k)}) - f(\tilde{M}^{(k)})|}{|S^{(k)} - \tilde{M}^{(k)}|_\phi}$$

$$|S^{(k)} - \tilde{M}^{(k)}|_\phi \le \frac{L([n^{\varepsilon/2}]}{L(n)} + \frac{L(n) - L([n^{\varepsilon/2}])}{2L(n)} \sup_{1+[n^{\varepsilon/2}]\le k\le n} |S^{(k)} - \tilde{M}^{(k)}|_\phi$$

$$\le \frac{\varepsilon}{2} + \frac{1}{2} \sup_{1+[n^{\varepsilon/2}]\le k\le n} |S^{(k)} - \tilde{M}^{(k)}|_\phi$$

whence

$$\mathbb{P}\{d_\phi(W_n, W_n^{\tilde{M}}) > \varepsilon\} \le \mathbb{P}\{\sup_{1+[n^{\varepsilon/2}]\le k\le n} |S^{(k)} - \tilde{M}^{(k)}|_\phi > \varepsilon\}.$$

Condition (4.5) implies that for every $\varepsilon > 0$ and $N > 0$, there exists k_0 such that for any $k \geq k_0$, $\mathbb{P}\{|S^{(k)} - \tilde{M}^{(k)}|_\phi > \varepsilon\} < k^{-\frac{2N}{\varepsilon}-2}$. Therefore,

$$\mathbb{P}\{d_\phi(W_n, W_n^{\tilde{M}}) > \varepsilon\} \leq \sum_{k=[n^{\varepsilon/2}]+1}^{n} \mathbb{P}\{|S^{(k)} - \tilde{M}^{(k)}|_\phi > \varepsilon\} \leq$$

$$\sum_{k=[n^{\varepsilon/2}]+1}^{n} k^{-\frac{2N}{\varepsilon}} k^{-2} \leq n^{-N} \sum_{k=1}^{\infty} k^{-2} = cn^{-N}.$$

where c is a positive constant, and thus

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d_\phi(W_n, W_n^{\tilde{M}}) > \varepsilon\} = -\infty$$

so according to Lemma 4.1 the sequences W_n and $W_n^{\tilde{M}}$ are LDP equivalent. ■

Define for Q in $\mathcal{M}_1(\mathcal{C}[0,\infty))$ the function

$$I(Q) := \begin{cases} \lim_{a\to\infty} \frac{1}{2\log a} h\left(Q \circ |^{-1}_{[\frac{1}{a},a]} / W \circ |^{-1}_{[\frac{1}{a},a]}\right) & \text{if } Q \text{ is } \theta\text{-invariant} \\ \infty & \text{otherwise} \end{cases} \tag{4.6}$$

Theorem 4.3 *Let ϕ be the continuous function defined in (4.1) and the set $\mathcal{C}_\phi$ defined in (4.2). Assume that $\inf_{t\in\mathbf{R}_+} \phi(t) \geq C > 0$. Then the sequence (W_n), with $W_n = \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k}\delta_{S^{(k)}}$, satisfies the large deviation principle with constants $(\log n)$ and rate function $I|_{\mathcal{M}_1(\mathcal{C}_\phi)}$. That is, for any Borel set $A \subseteq \mathcal{M}_1(\mathcal{C}_\phi)$,*

$$\begin{aligned} -\inf_{A^\circ} I &\leq \liminf_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{W_n \in A\} \\ &\leq \limsup_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{W_n \in A\} \leq -\inf_{\bar{A}} I \end{aligned}$$

Proof: Using the martingale decomposition theorem, $S_t^{(n)} = \tilde{M}_t^{(n)} + R_t^{(n)}$, where $\tilde{M}_t^{(n)}$ is a L^2-martingale and invoking a LDP result for martingales ([9]), it remains to prove that W_n and $W_n^{\tilde{M}}$ are LDP equivalent. According to Lemma 4.2, this is equivalent to verifying the condition (4.5). We have:

$$\mathbb{P}\{|S^{(n)} - \tilde{M}^{(n)}|_\phi > \varepsilon\} = \mathbb{P}\{\sup_{t\in\mathbf{R}_+} \frac{|S_t^{(n)} - \tilde{M}_t^{(n)}|}{\phi(t)} > \varepsilon\} = \mathbb{P}\{\sup_{t\in\mathbf{R}_+} \frac{|R_t^{(n)}|}{\phi(t)} > \varepsilon\sigma\sqrt{n}\} \leq \mathbb{P}\{\sup_{t\in\mathbf{R}_+} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{n}\}$$

In the proof of the Theorem 2.4 we showed that there exists a sequence a_n, $\lim_{n\to\infty} a_n = \infty$, such that

$$\mathbb{P}\{\sup_{t\in[0,1]} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{n}\} \leq n^{-a_n}.$$

Similarly, one can get for any $N > 0$,

$$\mathbb{P}\{\sup_{t\in[0,N]} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{n}\} \leq n^{-a_n}.$$

Since the sequence $A_N = \{\sup_{t\in[0,N]} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{n}\}$ is increasing, by letting $N \to \infty$ we get

$$\mathbb{P}\{\sup_{t\in[0,\infty)} |R_t^{(n)}| > C\varepsilon\sigma\sqrt{n}\} \leq n^{-a_n}$$

and the condition (4.5) follows.

■

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