Functional large deviation principle for a time-changed Brownian motion

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Abstract

A large deviation principle for a normalized time-changed Brownian motion is obtained using a weak convergence approach. The time-change stems from the study of parabolic Cauchy problems with state-dependent intensity coefficients. Using the duality weak convergence - large deviations, we prove a large deviation principle for the superposition between the time-changed Brownian motion and the inverse process of the additive functional that determines the time change.

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1. Introduction

Let's consider the following parabolic Cauchy problem,

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \lambda(x)\Delta u(t,x), & t \geq 0,\, x \in \mathbb{R}^d, \\ u(0,x) = f(x), & t = 0,\, x \in \mathbb{R}^d, \end{cases} \tag{1}$$

with Δ being the standard Laplace operator, and $\lambda(x)$ the intensity coefficient.

Assume that one can construct a Markov process $X = \{X_t,\, t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, starting from the point $x \in \mathbb{R}^d$, with respect to the natural filtration $\{\mathcal{F}_t^X,\, t \geq 0\}$, $\mathcal{F}_t^X = \sigma(X_s,\, s \leq t)$, that has

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the *infinitesimal generator* $\lambda(x)\Delta$. Then, the solution of the Cauchy problem (1) is represented as

$$u(t,x) = \mathbb{E}[f(X_t) \,|\, X_0 = x] = P_t f(x),$$

where $\{P_t\}_{t\geq 0}$ is the Markov semigroup associated to the infinitesimal generator $\lambda(x)\Delta$.

There are situations when directly constructing a Markov semigroup that corresponds to an infinitesimal generator L is not feasible, thus new approaches need to be investigated. There are few known methods of constructing Markov semigroups ([1]), one of them being the *time change* method. The time change from t to $c\,t$, for some constant $c > 0$, of a Markov semigroup, corresponds to a change of the infinitesimal generator from L to $c\,L$. This transformation is very convenient for normalizing the infinitesimal generators, while a change of the infinitesimal generator L into $\lambda(x)L$ would require a random rate function that depends on the position of the process. Note that the *carré du champ* operator, defined as the bilinear map $\Gamma(f,g) = \frac{1}{2}[L(fg) - fL(g) - gL(f)]$, has the property that $\Gamma(f) = \Gamma(f,f) \geq 0$ on the domain of L. Since the transformed carré du champ operator is $\lambda(x)\Gamma$, the coefficient $\lambda(x)$ should be positive. Moreover, to ensure the continuity of the associated Markov process, we will restrict to the case of strictly positive bounded measurable functions, $0 < a \leq \lambda(x) \leq b < \infty$. The time-change method is described next.

Let $\{X_t,\, t \geq 0\}$ be a Markov process starting from a fixed point $x \in \mathbb{R}^d$, with infinitesimal generator L. Define the additive functional,

$$S_X(t,\omega) = \int_0^t \frac{ds}{\lambda(X_s(\omega))},$$

and its generalized right-continuous inverse process $\{\tau_t,\, t \geq 0\}$,

$$\tau_t(\omega) = \inf\{u > 0 :\, S_X(u,\omega) > t\},\ \inf\emptyset = +\infty.$$

Since $\lambda(x)$ is assumed uniformly bounded, the additive functional $S_X(t)$ (ω fixed) and the time change τ_t are continuous and increasing functions ([7], section 3.4), such that $S_X(\tau_t) = t$. Additionally, S_X is a bijective map from $\mathbb{R}_+$ to $\mathbb{R}_+$, with derivative $S'_X(t) = \frac{1}{\lambda(X_t)}$, and for any $t \geq 0$, $\tau'_t = \frac{1}{S'(\tau_t)} = \lambda(X_{\tau_t})$. Since the process S_X is adapted to the natural filtration $\mathcal{F}^X$, the time change τ_t is a stopping time with respect to this filtration. Due to the strong Markov property applied to the stopping times τ_t, the time-changed process $\hat{X}_t := X_{\tau_t}$ becomes a Markov process. We show that its infinitesimal generator is $\lambda(x)L$.

Indeed, using Itô's formula, one derives $df(X_t) = dM_t^f + Lf(X_t)dt$, where $\{M_t^f,\, t \geq 0\}$ is a local martingale with respect to the filtration $\{\mathcal{F}_t^X,\, t \geq 0\}$. By composition rule, $df(X_{\tau_t}) = \tau_t'[dM_{\tau_t}^f + Lf(\tau_t)dt]$, and thus $df(\hat{X}_t) = d\hat{M}_t^f + \lambda(\hat{X}_t)Lf(\hat{X}_t)$, where $\hat{M}_t^f$ is a local martingale with respect to the filtration $\mathcal{F}_{\tau_t}^X := \{A \in \mathcal{F} :\, A \cap \{\tau_t \leq u\} \in \mathcal{F}_u\}$.

Specialize X_t to be the standard Brownian motion B_t with infinitesimal generator $\Delta = \frac{1}{2}\frac{d^2}{dx^2}$. The time-changed process $\hat{X}_t = B_{\tau_t}$ becomes a diffusion process and a martingale with respect to the filtration $\mathcal{F}_{\tau_t}$, represented as

$$\hat{X}_t = \hat{X}_0 + \int_0^t \sqrt{\lambda(\hat{X}_s)}\, d\tilde{B}_s, \tag{2}$$

for some Wiener process $\tilde{B}$. Thus, starting with the infinitesimal generator $\Delta = \frac{1}{2}\frac{d^2}{dx^2}$, and performing the random time change, one gets an infinitesimal generator of the form $L = \lambda(x)\Delta$, with $\lambda(x)$ a uniformly bounded coefficient. Moreover, if a further Girsanov transformation is performed, the infinitesimal generator changes into $L = \frac{1}{2}\lambda(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$, with bounded measurable coefficients $b(x)$ and $\lambda(x)$, thus representing a larger class of diffusions that do not require Lipschitz coefficients.

Probabilistic approaches to Cauchy problems with sufficiently smooth coefficients (such as Lipschitz conditions) have long been studied, and Freidlin-Wentzell theory ([6]) covers an extensive analysis of large deviations for random perturbations of such systems. Physical examples that exhibits non-Lipschitz singularities motivated the study of Cauchy problems with irregularities and singularities. Viscosity solutions methods are often used to overcome such issues, but not easily manageable. As an alternative approach, the time change method was used by Kondratiev et all in [8] to study the asymptotic behavior of the stochastic processes associated with such systems. The setup is the following.

Let $\mathcal{C}[0,T]$ be the space of continuous functions and $\mathcal{C}^{\uparrow}[0,T]$ the space of continuous and increasing functions. Consider the sequence of stochastic processes $\{(\nu_n, \zeta_n)\}_{n\geq 1}$, defined on the product space $\mathcal{C}^{\uparrow}[0,T] \times \mathcal{C}[0,T]$, endowed with the corresponding σ-fields generated by the uniform topology, as follows:

$$\nu_n(t) = \frac{1}{n}\tau_{nt}, \quad Z_n(t) = \frac{1}{\sqrt{n}}B_{nt}, \quad \zeta_n(t) = Z_n(\nu_n(t)) = \frac{1}{\sqrt{n}}B_{\tau_{nt}}. \tag{3}$$

The functional limit theorems for the normalized time-changed Brownian motion have been studied in [8] under different scenarios regarding the intensity coefficient λ, including the case in which the limits, $\lim_{x\to\pm\infty}\lambda(x)=a_\pm$, with $a_\pm>0$, exist. In this case, it has been proved (Theorem 3.2.1 in [8]) that for $t\in[0,T]$,

$$\frac{B_{\tau_{nt}}}{\sqrt{n}} \Rightarrow W(\eta^{-1}(t)), \quad \text{as} \quad n\to\infty, \tag{4}$$

where W is a Wiener process and

$$\eta(t)=\int_0^t \gamma(W(s))\,ds, \quad \gamma(x)=\frac{1}{a_+}\mathbb{1}_{(0,\infty)}(x)+\frac{1}{a_-}\mathbb{1}_{(-\infty,0)}(x). \tag{5}$$

In this paper we study a large deviation principle (LDP) for the normalized time-changed Brownian motion under the conditions described above. Using the duality between the large deviation theory and the weak convergence theory ([4], [5]), we develop a similar argument, based on superposition, as in [8].

2. Large deviation principle

The theory of large deviations is devoted to estimating normalizations of $\log \mathbb{P}(A_n)$ for sequences of events with asymptotically vanishing probability. The standard formulation, due to S.R.S. Varadhan ([14], 1966), is that the sequence of random variables $\{X_n\}_{n\geq 1}$ with values in a complete separable metric space $(E,\mathcal{B}_E)$ satisfies the *large deviation principle* if there exists a lower semicontinuous function $I:E\to[0,\infty]$ such that:

(i) for each open set $G\in E$,

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\{X_n\in G\}\geq -\inf_{x\in G}I(x);$$

(ii) for each closed set $C\in E$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\{X_n\in C\}\leq -\inf_{x\in C}I(x);$$

(iii) for each $a\in[0,\infty)$, $\{x:\ I(x)\leq a\}$ is a compact set in E.

The function I satisfying (i) and (ii) is called the *rate function*, and if in addition (iii) holds, it is called a *good* rate function for the large deviation principle. Therefore, in the literature, it is said that a sequence of random variables satisfies a LDP with rate function I, if (i) and (ii) hold, and it satisfies a *full* LDP if the rate function is good. Also, it is said that the sequence of random variables satisfies a *weak* LDP with rate function I if (ii) holds for compact sets. In the sequel, by LDP we mean a full LDP.

Let $E = \mathcal{C}[0,T])$ be the space of continuous functions on $[0,T]$, endowed with the uniform norm, and $\mathcal{C}_0[0,T]$ the space of continuous functions φ, satisfying $\varphi(0) = 0$. Denote by $H_0^2[0,T]$ the space of absolutely continuous functions $\varphi \in \mathcal{C}_0[0,T]$ with derivatives $\dot{\varphi} \in L^2[0,T]$. The celebrated functional large deviation principle for the Brownian motion, known as Schilder's Theorem ([14]), states that the sequence of stochastic processes $\{X_n\}_{n\geq 1}$, with $X_n(t) = \frac{1}{\sqrt{n}} B_t$, satisfies the LDP with the rate function $I : \mathcal{C}[0,T] \to [0,\infty]$,

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \dot{\varphi}(t)^2 \, dt, & \text{if} \quad \varphi \in H_0^2[0,T] \\ \infty, & \text{otherwise.} \end{cases} \tag{6}$$

A large deviation principle for time-changed Gaussian processes has been obtained in [11], with the time-change derived from a *subordination*. Subordination is a method of producing new Markov semigroups through time averaging, and in this case the process and the time change are independent. For large deviations of tempered subordinators and their inverses we refer to [9]. Another example, a large deviation principle for renewal processes and superpositions of independent renewal processes is given in [13], where a LDP for a sequence of stochastic processes leads to an LDP for the associated first-passage-time and inverse processes.

Here, the time change τ is *not* independent of the Brownian motion B, and moreover, the time-changed B_{τ_t} is *not* a square-integrable martingale. Therefore, a superposition argument as in [8] is natural. We will study the large deviation principle for the couple processes $\{(\nu_n, \zeta_n)\}_{n\geq 1}$ defined on the product space $\mathcal{C}^{\uparrow}[0,T] \times \mathcal{C}[0,T]$ and then derive the LDP for the marginal processes $\{\zeta_n\}_{n\geq 1}$ using the *contraction principle*. This principle states that the LDP is preserved under continuous mapping transformations. That is, if the sequence of stochastic processes $\{X_n\}_{n\geq 1}$ obeys an LDP with rate function I and if f is a continuous function, then $\{f(X_n)\}_{n\geq 1}$ obeys an LDP with rate function $I'(y) = \inf_{\{x:f(x)=y\}} I(x)$. A slight generalization ([15], [12]), which is called *extended contraction principle*, is stated as follows:

Theorem 2.1. *If the sequence of stochastic processes $\{X_n\}_{n\geq 1}$ obeys the LDP with rate function I and if $\{f_n\}_{n\geq 1}$ is a sequence of measurable functions and f is a continuous function on the restriction sets $\{x : I(x) \leq a\}$, $a \geq 0$, and if $f_n(x_n) \to f(x)$ as $n \to \infty$ for all sequences x_n such that $x_n \to x$ as $n \to \infty$, for all x for which $I(x) < \infty$, then the sequence $\{f_n(X_n)\}_{n\geq 1}$ obeys the LDP with rate function,*

$$I'(y) = \inf_{\{x:\, f(x)=y\}} I(x). \tag{7}$$

The duality between the weak convergence and the large deviation theory has been extensively employed (see e.g. [4], [5]). For example, an analogue of Prohorov's theorem where the *tightness* condition and the finite dimensional convergence lead to the weak convergence of stochastic processes ([2]), is that the *exponential tightness* and a *weak* large deviation principle determine a large deviation principle with a good rate function (see e.g. Puhalskii ([10, 12]).

Definition 2.2. *A sequence of random variables $\{X_n\}_{n\geq 1}$ on a metric space E is said to be exponentially tight with speed $\{r_n\}_{n\geq 1}$ if for any $N < \infty$, there exists a compact set $K_N \subset E$, such that*

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mathbb{P}(X_n \notin K_N) \leq -N.$$

Next we state a criteria for exponential tightness that was derived by A.A. Puhalskii ([12]) for stochastic processes on metric spaces.

Theorem 2.3. *A sequence of stochastic processes $\{X_n\}_{n\geq 1}$ is exponentially tight in $\mathcal{C}[0,T]$ with speed function $\{r_n\}$ if and only if the following conditions hold:*

(i) $\lim_{A\to\infty} \limsup_{n\to\infty} \frac{1}{r_n} \log \mathbb{P}\{|X_n(0)| > A\} = -\infty$*;*

(ii) $\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t\in[0,T]} \frac{1}{r_n} \log \mathbb{P}\{\sup_{s\in[t,t+\delta]} |X_n(s) - X_n(t)| > \epsilon\} = -\infty$ *for any* $T > 0$*, and* $\epsilon > 0$*.*

For a more general characterization of the exponential tightness, we refer to [5], Theorem 4.1.

Let us consider the following assumptions:

A1. λ is measurable such that $1/\lambda$ is locally integrable;

A2. there exists two positive numbers $a_\pm$ such that $\lim_{x\to\pm\infty}\lambda(x) = a_\pm$;

A3. λ is uniformly bounded, $0 < c \leq \lambda(x) \leq d$.

Proposition 2.4. *Under the assumptions A1−A3, the sequence of processes* $\{\zeta_n\}_{n\geq 1}$ *defined as* $\zeta_n(t) = \frac{1}{\sqrt{n}} B_{\tau_{nt}}$ *for* $t \in [0,T]$, *is exponentially tight on* $\mathcal{C}[0,T]$, *for any* $T > 0$.

Proof: We need to check the conditions (i) and (ii) of Theorem 2.3. Condition (i) is obviously true since $\zeta_n(0) = 0$. To prove (ii), let $T > 0$ and $\varepsilon > 0$ be fixed. For $\mu > 0$ be arbitrarily chosen, we have:

$$\begin{aligned}
&\mathbb{P}\{\sup_{s\in[t,t+\delta]} |\zeta_n(s) - \zeta_n(t)| > \varepsilon\} = \mathbb{P}\{\sup_{s\in[t,t+\delta]} \frac{1}{\sqrt{n}}|\hat{X}_{ns} - \hat{X}_{nt}| > \varepsilon\} \\
&= \mathbb{P}\{\sup_{s\in[t,t+\delta]} \exp(\mu\sqrt{n}|\hat{X}_{ns} - \hat{X}_{nt}|) > \exp(n\mu\varepsilon)\} \\
&\leq \exp(-n\mu\varepsilon)\mathbb{E}[\exp(\mu\sqrt{n}|\hat{X}_{n(t+\delta)} - \hat{X}_{nt}|)],
\end{aligned}$$

by Doob's martingale inequality.

Using the martingale decomposition (2) for the time-changed process, we write for some Wiener process $\tilde{B}_s$,

$$\begin{aligned}
&\mathbb{P}\{sup_{s\in[t,t+\delta]}|\zeta_n(s) - \zeta_n(t)|>\varepsilon\} \leq \exp(-n\mu\varepsilon)\mathbb{E}\left[\exp\left(\mu\sqrt{n}\int_{nt}^{n(t+\delta)} \sqrt{\lambda(\hat{X}_s)}d\tilde{B}_s\right)\right] \\
&= \exp(-n\mu\varepsilon)\mathbb{E}\left[\exp\left(\frac{\mu^2 n}{2}\int_{nt}^{n(t+\delta)} \lambda(\hat{X}_s))\,ds\right)\right] \\
&= \exp(-n\mu\varepsilon)\mathbb{E}\left[\exp\left(\frac{\mu^2 n^2}{2}\int_{t}^{t+\delta} \lambda(\hat{X}_{ns}))\,ds\right)\right].
\end{aligned}$$

Since $\lambda(x)$ is bounded, by the dominated convergence theorem, when $\delta \to 0$,

$$\limsup_{n\to\infty}\limsup_{\delta\to 0}\sup_{t\in[0,T]} \frac{1}{n}\log\mathbb{P}\{\sup_{s\in[t,t+\delta]}|\zeta_n(s) - \zeta_n(t)| > \varepsilon\} \leq -\mu\varepsilon,$$

for any $\mu > 0$ arbitrarily chosen, which proves condition (ii) of the exponential tightness. ■

Proposition 2.5. *Under the assumptions A1−A3, the sequence of processes* $\{S_B^{(n)}\}_{n\geq 1}$*, where* $S_B^{(n)}(t) = \frac{1}{n}S_B(nt)$ *for* $t \in [0,T]$*, satisfies the LDP on* $\mathcal{C}^{\uparrow}[0,T]$ *with rate function,*

$$I'(\psi) = \begin{cases} \frac{1}{2}\int_0^T \dot{\varphi}(t)^2\, dt, & \text{if } \varphi \in H_0^2[0,T] \text{ such that } \int_0^t \gamma(\varphi(u))du = \psi(t) \\ \infty, & \text{otherwise,} \end{cases} \tag{8}$$

where γ *is defined in (5).*

Proof: We first show the exponential tightness. As the condition (i) of Theorem 2.3 is obviously true, we will prove (ii). Let $T > 0$ and $\varepsilon > 0$ be fixed. For some Wiener process $\tilde{B}$ and $\mu > 0$ arbitrarily chosen, using the fact that $\lambda(x) \geq c$ we will get,

$$\begin{aligned}
&\mathbb{P}\{\sup_{s\in[t,t+\delta]}|S_B(ns) - S_B(nt)| > n\varepsilon\} = \mathbb{P}\{\sup_{s\in[t,t+\delta]}|\int_{nt}^{ns}\frac{ds}{\lambda(B_s)}| > n\varepsilon\} \leq \\
&\mathbb{P}\left\{\int_{nt}^{n(t+\delta)}\frac{ds}{\lambda(B_s)} > n\varepsilon\right\} = \mathbb{P}\left\{\int_t^{t+\delta}\frac{ds}{\lambda(\sqrt{n}\tilde{B}_s)} > \varepsilon\right\} \\
&\leq \exp(-n\mu\varepsilon)\mathbb{E}\left[\exp\left(n\mu\int_t^{t+\delta}\frac{ds}{\lambda(\sqrt{n}\tilde{B}_s)}\right)\right] \leq \exp(-n\mu\varepsilon)\exp\left(\frac{n\mu\delta}{c}\right).
\end{aligned}$$

By taking $\delta \to 0$, and then $n \to \infty$, it follows that

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\sup_{t\in[0,T]}\frac{1}{n}\log\mathbb{P}\{\sup_{s\in[t,t+\delta]}|S_B^{(n)}(s) - S_B^{(n)}(t)| > \varepsilon\} \leq -\mu\varepsilon,$$

which proves the exponential tightness of the process.

Note that

$$S_B^n(t) = \frac{1}{n}\int_0^{nt}\frac{ds}{\lambda(B_s)} = \frac{1}{n}\int_0^t\frac{ndu}{\lambda(B_{nu})} = \int_0^t\frac{du}{\lambda(\sqrt{n}W_u)} = \int_0^t\frac{du}{\lambda(nB_u^n)},$$

where $B_u^n = \frac{1}{\sqrt{n}}W_u$ satisfies the LDP with rate function (6).

Let $\{\varphi_n\}_{n\geq 1}$ be a sequence in $\mathcal{C}[0,T]$ such that $\varphi_n \to \varphi$ with respect to the uniform topology. For any $t \in [0,T]$, let $f_n(\varphi_n)(t) = \int_0^t \frac{du}{\lambda(\sqrt{n}\varphi_n(u))}$. Since $\lambda(x) \geq c > 0$ and $\lim_{x\to\pm\infty}\lambda(x) = a_\pm$, we apply the Lebesgue's dominated convergence theorem and get

$$\begin{aligned}\lim_{n\to\infty} f_n(\varphi)(t) &= \int_0^t \lim_{n\to\infty} \frac{du}{\lambda(\sqrt{n}\varphi_n(u))} \\ &= \int_0^t \left[\frac{1}{a_+}1\!\!1_{(0,\infty)}(\varphi(u)) + \frac{1}{a_-}1\!\!1_{(-\infty,a)}(\varphi(u))\right] du := f(\varphi)(t).\end{aligned}$$

Due to the extended contraction principle (Theorem 2.1), the sequence of stochastic processes $\{S_B^n := f_n(B^n),\, n \geq 1\}$ obeys the large deviation principle on $\mathcal{C}^{\uparrow}[0,T]$ with rate function

$$I'(\psi) = \inf_{\{\varphi : \int_0^t \gamma(\varphi(u)du = \psi(t)\}} I(\varphi).$$

Note that for any $t, s > 0$, $\psi(t+s) - \psi(t) > 0$, so $\psi(t) = \int_0^t \gamma(\varphi(u)du$ is a continuous and strictly increasing function, therefore its inverse exists and is continuous, and thus the sequence $\{S_B^n\}_{n\geq 1}$ satisfies the LDP on $\mathcal{C}^{\uparrow}[0,T]$ with rate function given in (8). ■

Proposition 2.6. *Under the assumptions A1−A3, the sequence of processes $\{\nu_n\}_{n\geq 1}$, with $\nu_n(t) = \frac{1}{n}\tau_{nt}$, satisfies the LDP on $\mathcal{C}^{\uparrow}[0,T]$ with rate function*

$$I_1(\phi) = \begin{cases} \frac{1}{2}\int_0^T \dot{\varphi}(t)^2\,dt, & \text{if } \varphi \in H_0^2[0,T] \text{ such that } \int_0^t \gamma(\varphi(u))\,du = \phi^{-1}(t) \\ \infty, & \text{otherwise.} \end{cases} \tag{9}$$

with γ defined in (5).

Proof:

According to Theorem 3.1 in [13], a large deviation principle for an inverse process with linear scalings is obtained from the LDP of the given process. Therefore the sequence $\{\nu_n\}_{n\geq 1}$, where $\nu_n(t) = \frac{1}{n}\tau(nt)$, obeys an LDP on $\mathcal{C}^{\uparrow}[0,T]$ with rate function $I_1(\phi) = I'(\phi^{-1})$. Since I' is a good rate function, such is I_1, making the sequence $\{\nu_n\}_{n\geq 1}$ exponentially tight. ■

Proposition 2.7. *Let $Z_n(t) = \frac{1}{\sqrt{n}}B_{nt}$ and $\{x_n\}_{n\geq 0}$ be a sequence in $\mathcal{C}^{\uparrow}[0,T]$ such that $x_n \to x$ in $\mathcal{C}[0,T]$. Then the sequence of stochastic processes $\{Z_n(x_n)\}_{n\geq 1}$ satisfies a large deviation principle with rate function*

$$I_2(\Phi \mid x) = \begin{cases} \frac{1}{2}\int_0^T (\dot{\varphi}(t))^2 dt, & \text{if } \varphi \in H_0^2[0,T] \text{ such that } \varphi \circ x = \Phi \\ \infty, & \text{otherwise.} \end{cases} \tag{10}$$

Proof:

We show that the sequences of stochastic processes $\{Z_n(x_n)\}_{n\geq 1}$ and $\{Z_n(x)\}_{n\geq 1}$ are LDP equivalent (which means that if one obeys an LDP with rate function I, the same is true for the other). By Lemma 3.13 in [5], it suffices to prove that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{\sup_{t\in[0,T]} |Z_n(x_n(t)) - Z_n(x(t))| > \varepsilon\} = -\infty.$$

Indeed, let $\varepsilon > 0$ be fixed. For some $\mu > 0$ arbitrarily chosen we have:

$$\begin{aligned}
\mathbb{P}\{\sup_{t\in[0,T]} |Z_n(x_n(t)) - Z_n(x(t))| > \varepsilon\} &= \mathbb{P}\{\sup_{t\in[0,T]} |B(nx_n(t)) - B(nx(t))| > \sqrt{n}\varepsilon\} \\
&= \mathbb{P}\{\mu\sqrt{n} \sup_{t\in[0,T]} |B(nx_n(t)) - B(nx(t))| > \mu n\varepsilon\} \\
&= \mathbb{P}\{\mu n \sup_{t\in[0,T]} |B(x_n(t)) - B(x(t))| > \mu n\varepsilon\} \\
&\leq \exp(-n\mu\varepsilon)\mathbb{E}\left[\exp(\mu n \sup_{t\in[0,T]} |B(x_n(t)) - B(x(t))|)\right] \\
&\leq \exp(-n\mu\varepsilon)\exp\left(\mu n \sup_{t\in[0,T]} c|x_n(t) - x(t)|^{\alpha}\right),
\end{aligned}$$

where $c > 0$ and $0 < \alpha < \frac{1}{2}$, due to the α-Holder continuity of the Brownian motion.

As $x^n \to x$ in $\mathcal{C}^{\uparrow}[0,T]$, $\lim_{n\to\infty} \sup_{t\in[0,T]} |x^n(t) - x(t)| = 0$. Therefore, there exists $n \geq n_0(\delta)$ such that $\sup_{t\in[0,T]} |x^n(t) - x(t)| < \delta$ and such that $\delta < \left(\frac{\varepsilon}{c}\right)^{1/\alpha}$. Thus,

$$\frac{1}{n} \log \mathbb{P}\{\sup_{t\in[0,T]} |Z_n(x^n(t)) - Z_n(x(t))| > \varepsilon\} \leq -\mu(\varepsilon - c\delta^{\alpha}),$$

for any $\mu > 0$ and $n \geq n_0$, so

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{\sup_{t\in[0,T]} |Z_n(x^n(t)) - Z_n(x(t))| > \varepsilon\} = -\infty.$$

For fixed $x(t) \in \mathcal{C}^{\uparrow}[0,T]$, $\sup_{t\in[0,T]} x(t) = x(T)$ and

$$F : \mathcal{C}[0, x(T)] \to \mathcal{C}[0,T],\ F(\Phi) = \Phi \circ x,$$

is continuous. Therefore, by the contraction principle, the sequence $\{Z_n(x)\}_{n\geq 1}$ satisfies the LDP with rate function $I_2(\Phi \,|\, x)$ on $\mathcal{C}[0, x(T)]$ given in (10). ■

Next, we consider the construction of a rate function on the product space. If $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ are two Polish spaces with their associated Borel σ-fields, and $\{\mu_{1n}\}_{n\geq 1}$ a sequence of probability measures on $(E_1, \mathcal{E}_1)$, let $\{\nu_n(x_1, B_2)\}_{n\geq 1}$ be a sequence of probability transition functions defined on $E_1 \times \mathcal{E}_2$. Then, a sequence a probability measure $\{\mu_n\}_{n\geq 1}$ on the product space $E_1 \times E_2$ is represented as,

$$\mu_n(B_1 \times B_2) = \int_{B_1} \nu_n(x_1, B_2) d\mu_{1n}(x_1), \tag{11}$$

for any $B_i \in \mathcal{E}_i$, $i = 1, 2$.

The following definition was introduced by Chaganty in [3] to provide a large deviation criteria on a product space.

Definition 2.8. *The sequence of probability transition functions $\{\nu_n(x_1, \cdot)\}$, $x_1 \in E_1$, satisfies the (LDP) continuity condition in x_1 with rate function $J(x_1, x_2)$ if*

(i) for each $x_1 \in E_1$, $J(x_1, \cdot)$ is a rate function on E_2;

(ii) for any sequence $\{x_{1n}\}$ in E_1 such that $x_{1n} \to x_1$, the sequence of measures $\{\nu_n(x_{1n}, \cdot)\}$ on E_2 obeys the LDP with rate function $J(x_1, \cdot)$;

(iii) $J(x_1, x_2)$ is lower semi-continuous as a function of (x_1, x_2).

To prove the large deviation principle on the product space we use Theorem 2.3 from [3], which states:

Theorem 2.9. *Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two Polish spaces with their associated Borel σ-fields. Suppose that the sequence of probability measures $\{\mu_n\}$ defined in (11) is exponentially tight on the product space $E_1 \times E_2$, and that the sequence of probability transition functions $\{\nu_n(x_1, B_2)\}$ satisfies the (LDP) continuity condition with rate function $J(x_1, x_2)$. If $\{\mu_{1n}\}$ obeys LDP with rate function $I_1(x_1)$, then the sequence $\{\mu_n\}$ satisfies a large deviation principle with rate function*

$$I(x_1, x_2) = I_1(x_1) + J(x_1, x_2).$$

Consequently, the marginal sequence of measures $\{\mu_{2n}\}$ on E_2 also obeys the LDP with rate function

$$I_2(x_2) = \inf_{x_1 \in E_1} [I_1(x_1) + J(x_1, x_2)].$$

We can now state the main result.

Theorem 2.10. *Let B_{τ_t} be the time-changed Brownian motion associated to the infinitesimal generator $\lambda(x)\Delta$, where Δ is the Laplace operator and $\lambda : \mathbb{R} \to (0,\infty)$ is a measurable function satisfying the following assumptions:*

(A1) λ is measurable such that $1/\lambda$ is locally integrable;

(A2) there exists two positive numbers $a_\pm$ such that $\lim_{x\to\pm\infty} \lambda(x) = a_\pm$;

(A3) λ is uniformly bounded, $0 < c \le \lambda(x) \le d$.

Then the sequence of normalized time-changed processes $\{\zeta_n\}_{n\ge1}$, where $\zeta_n(t) = \frac{1}{\sqrt{n}} B_{\tau_{nt}}$ for $t \in [0,T]$, satisfies the large deviation principle with rate function $I : \mathcal{C}[0,T] \to [0,\infty]$,

$$I_2(\Phi) = \inf_{\phi\in\mathcal{C}^\uparrow[0,T]} [I_1(\phi) + I_2(\Phi \,|\, \phi)],$$

with $I_1(\phi)$ and $I_2(\Phi \,|\, \phi)$ given in equations (9) and (10), respectively.

Proof:
We apply Theorem 2.9 to the sequence of processes $\{(\nu_n,\zeta_n)\}_{n\ge1}$. The exponential tightness follows from [5] (Lemma 3.6), stating that a sequence of probability measures on the product space is exponentially tight if and only if the marginal distributions are exponential tight, provided that each individual probability measure is tight. We know that any probability measure defined on a Polish space is tight (Theorem 1.3 in [2]). Accordingly, since $\mathcal{C}[0,T]$ equipped with the supremum norm topology is a Polish space, the exponential tightness of the couple processes follows from the exponential tightness of the marginal processes. Since $\{\zeta_n\}_{n\ge1}$ is exponentially tight on $\mathcal{C}[0,T]$ (Proposition 2.4), and $\{\nu_n\}_{n\ge1}$ is exponentially tight on $\mathcal{C}^\uparrow[0,T]$ (from the proof of Proposition 2.6), then the sequence $\{(\nu_n,\zeta_n)\}_{n\ge1}$ is exponentially tight on $\mathcal{C}^\uparrow[0,T] \times \mathcal{C}[0,T]$.

Next, we prove the (LDP) continuity conditions for the transition measures. That is, show that the rate function $J(\phi,\Phi) := I_2(\Phi \,|\, \phi)$ satisfies the conditions (i)-(iii) in Definition 2.8. In Proposition 2.7 we proved that the conditions (i) and (ii) hold true. It only remains to verify condition (iii), which is the lower semicontinuity property of $J(\phi,\Phi)$.

Indeed, if we assume that $(\phi^n, \Phi^n) \to (\phi, \Phi)$, then, when $J(\phi^n, \Phi^n) < \infty$, using Fatou's lemma, we get

$$\begin{aligned}\liminf_{n\to\infty} J(\phi^n, \Phi^n) &= \liminf_{n\to\infty} \frac{1}{2}\int_0^T \frac{d}{dt}[\Phi^n((\phi^n)^{-1})]dt \\ &\geq \frac{1}{2}\int_0^T \lim_{n\to\infty} \frac{d}{dt}[\Phi^n((\phi^n)^{-1})dt = J(\Phi, \phi).\end{aligned}$$

Thus, the sequence of couple processes $\{(\nu_n, \zeta_n)\}_{n\geq 1}$ obeys the LDP with rate function $I(\phi, \Phi) = I_1(\phi) + J(\phi, \Phi)$. Consequently, due to the contraction principle, the sequence of process $\{\zeta_n\}_{n\geq 1}$ obeys the LDP with rate function (10). ■

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