LARGE DEVIATIONS APPLICATION TO EXIT TIMES FOR SWITCHED MARKOV PROCESSES

ADINA OPRISAN[1], ANDRZEJ KORZENIOWSKI[2]
[1]DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, BARRY UNIVERSITY
MIAMI SHORES, FL 33161, USA
AOPRISAN@MAIL.BARRY.EDU
[2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT ARLINGTON
BOX 19408, ARLINGTON, TX 76019, USA
KORZENIOWSKI@UTA.EDU

Abstract: We study the effect of small perturbations on large time intervals for a family of stochastic additive functionals of Markov processes switched by jump Markov processes. The averaged limit process evolves deterministically on random time intervals according to the transition times of a stationary jump Markov process. Small perturbations essentially influence the behavior of the system and asymptotics of large deviations play an important role in analyzing it.

AMS Subject Classification: 60F10, 60H10, 60B10

Key Words: Large deviations, additive functionals, jump Markov processes, average approximation, stability, exit times.

1. Introduction

The purpose of this paper is to develop a variant of the classical Freidlin-Wentzell theory for a class of stochastic additive functionals switched by jump Markov processes in averaging approximation. Large deviation principle (LDP), exit time problem, stochastic stability for randomly perturbed dynamical systems and the averaging principle have been thoroughly studied by Freidlin and Wentzell in [2]. The stochastic stability and optimal control for evolutionary systems in averaging scheme for Markov and semi-Markov random evolutions, including Dirichlet problems for multiplicative operator functionals of Markov processes, were developed in [7]. For estimates of probabilities of hitting a boundary point for Poisson processes and jump Markov processes we refer the reader to [6].

The paper is organized as follows: in section 2 we introduce the stochastic additive functionals, averaging principle and large deviation principle. In section 3 we apply a useful quasi-potential method alternative to LDP, which plays an important role in the asymptotics of exit times. We first consider the jump Markov process which is uniformly ergodic on E, followed by generalization to uniform ergodicity for each partition E_k of the split space in section 4.

2. Preliminary results

2.1. **Average approximation theorem.** Let $(E, \mathcal{E})$ be a complete, separable metric space and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ right-continuous and complete. Let $\eta(t;x)$, $t \geq 0, x \in E$ be a process with locally independent increments and $\{x(t),\, t \geq 0; q(x), x \in E\}$ be a switching jump Markov process in E with jump intensity rates $q(x)$. The switching processes that describe the random changes in the evolution of the system, are considered in a split space $E = \cup_{k=1}^{N} E_k$, $E_k \cap E_{k'} = \emptyset$, $k \neq k'$ with non-communicating components, and having the ergodic property on each class E_k. By introducing the parameter $\varepsilon > 0$ one defines a jump Markov process on the split phase space with small transition probabilities between the states of the system and further merges the classes E_k, $k = 1, 2, \cdots, N$ into distinct states k, $1 \leq k \leq N$.

Consider the family of stochastic additive functionals $\xi^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$,

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\left(ds; x^\varepsilon(\frac{s}{\varepsilon})\right), \quad t \geq 0, \varepsilon > 0 \tag{2.1}$$

The infinitesimal generator of $\eta^\varepsilon(t;x)$ is

$$\mathbb{\Gamma}^\varepsilon(x)\varphi(u) = a^\varepsilon(u;x)\varphi'(u) + \varepsilon^{-1}\int[\varphi(u+\varepsilon v) - \varphi(u) - \varepsilon v\varphi'(u)]\Gamma_\varepsilon(u,dv;x)$$

with the drift velocity $a(u;x)$ in the Banach space $B^1(\mathbb{R}^d)$ of bounded and continuous functions subject to

$$a^\varepsilon(u;x) = a(u;x) + \theta^\varepsilon(u;x)$$

where $\theta^\varepsilon(u;x) \to 0$ as $\varepsilon \to 0$ uniformly on $(u;x)$, $\Gamma_\varepsilon(u,dv;x) \equiv \Gamma(u,dv;x)$ is independent of ε and the operator

$$\gamma^\varepsilon(x)\varphi(u) = \varepsilon^{-1}\int_{\mathbb{R}^d}[\varphi(u+\varepsilon v) - \varphi(u) - \varepsilon v\varphi'(u)]\Gamma(u,dv;x)$$

is negligible on $B^1(\mathbb{R}^d)$.

The switching Markov process $x^\varepsilon(t), t \geq 0$ is defined by the generator

$$Q^\varepsilon\varphi(x) = q(x)\int_E P^\varepsilon(x,dy)[\varphi(y) - \varphi(x)]$$

and satisfies the *Phase Merging Principle* (PMP) assumptions:

(PMP1) The stochastic kernel $P^\varepsilon(x,B)$ is represented by

$$P^\varepsilon(x,B) = P(x,B) + \varepsilon P_1(x,B)$$

where the stochastic kernel $P(x,B)$ of the embedded Markov chain corresponding to $x(t)$ is coupled with the splitting according to

$$P(x,E_k) = \mathbb{1}_k(x) := \begin{cases} 1, & x \in E_k, \\ 0, & x \notin E_k \end{cases}$$

and the perturbing operator $P_1(x,B)$ is a signed kernel which satisfies the conservative condition $P_1(x,E) = 0$.

(PMP2) The Markov supporting process $x(t)$, $t \geq 0$ on the state space $(E,\mathcal{E})$ is supposed to be uniformly ergodic in every class E_k, $1 \leq k \leq N$, with the stationary distribution $\pi_k(dx)$, $1 \leq k \leq N$, satisfying the following relations

$$\pi_k(dx)q(x) = q_k\rho_k(dx), \qquad q_k = \int_{E_k}\pi_k(dx)q(x),$$

$$\rho_k(B) = \int_{E_k}\rho_k(dx)P(x,B), \qquad \rho_k(E_k) = 1.$$

(PMP3) The average exit probabilities satisfy the following condition

$$\hat{p}_k := \int_{E_k}\rho_k(dx)P_1(x,E\backslash E_k) > 0, \qquad 1 \leq k \leq N.$$

The infinitesimal generator of the stochastic additive functional $\xi^\varepsilon(t)$ is

$$\mathbb{L}^\varepsilon = Q^\varepsilon + \mathbb{\Gamma}^\varepsilon. \tag{2.2}$$

Theorem 2.1. (Average approximation) [3] *The stochastic evolutionary system* $\xi^\varepsilon(t)$, $t \geq 0$ *defined by* (2.1) *converges weakly to the averaged stochastic system* $\hat{\xi}(t)$,

$$\xi^\varepsilon(t) \Rightarrow \hat{\xi}(t) \qquad as \qquad \varepsilon \to 0.$$

The limit process $\hat{\xi}(t)$, $t \geq 0$ *is defined by a solution of the evolutionary equation*

$$\frac{d}{dt}\hat{\xi}(t) = \hat{a}(\hat{\xi}(t); \hat{x}(t)), \qquad \hat{\xi}(0) = \xi(0) \tag{2.3}$$

where the averaged velocity is determined by

$$\hat{a}(u;k) = \int_{E_k} \pi_k(dx) a(u;x) \qquad 1 \leq k \leq N.$$

The limit process $\hat{\xi}(t)$ is a random dynamical system evolving deterministically on random time intervals $[T_i, T_{i+1})$, where $\{T_i\}_{i=1}^{N(T)}$ are the transition times of the stationary merged process $\hat{x}(t)$ and $N(T)$ the number of transitions on $[0, T]$.

2.2. **Dirichlet problem.**

Definition 2.2. $\vartheta \in \mathbb{R}^d$ is an asymptotically stable position of the averaged system (2.3) if for every neighborhood $\mathcal{E}_1$ of ϑ there exists a smaller neighborhood $\mathcal{E}_2 \subset \mathcal{E}_1$ of ϑ such that the trajectories of $\hat{\xi}_t(u)$ starting in $\mathcal{E}_2$ converge to ϑ without leaving $\mathcal{E}_1$ as $t \to \infty$.

Definition 2.3. Let $D \in \mathbb{R}^d$ be a bounded domain and ∂D be its smooth boundary. The domain D is attracted to ϑ if the trajectories $\hat{\xi}_t(u), u \in D$ converge to the equilibrium position ϑ without leaving D as $t \to \infty$.

Consider the Dirichlet Problem associated to the generator $\mathbb{L}^\varepsilon$ defined in (2.2) :

$$\begin{cases} \mathbb{L}^\varepsilon \varphi^\varepsilon(u;x) = 0 & \text{for } u \in D \subset \mathbb{R}^d \\ \varphi^\varepsilon(u;x) = f(u;x) & \text{for } u \in \partial D,\ x \in E \end{cases} \tag{2.4}$$

where D is a bounded open set and f is continuous.

The solution depends on the exit time of the stochastic additive functional from the bounded domain D, see [4] (more general, for multiplicative functionals analogue of Dynkin formula and solutions of Dirichlet problems in evolutionary systems with Markov or semi-Markov switching are given in [7])

$$\varphi^{\varepsilon}(u;x) = \mathbb{E}_{u,x}[f(\xi^{\varepsilon}(\tau^{\varepsilon});x^{\varepsilon}(\tau^{\varepsilon}))] \tag{2.5}$$

where $\tau^{\varepsilon} := \inf\{t : \xi^{\varepsilon}(t) \in \partial D\}$ and the expected value is taken under $\mathbb{P}^{\varepsilon}_{u,x}$, the measure on the path space corresponding to the generator $\mathbb{L}^{\varepsilon}$ that starts at $u \in D$, $x \in E$ at time 0.

As $\varepsilon \to 0$, $\mathbb{P}^{\varepsilon}_{u,x} \Rightarrow \mathbb{P}_{u,x}$ the degenerate measure at the solution of the averaged system. We are interested in the behavior of $\xi^{\varepsilon}(\tau^{\varepsilon})$ as $\varepsilon \to 0$. To study this exit time we make use of the large deviation principle for the family of stochastic additive functionals.

2.3. **Large deviations for additive functionals.** Based on the average approximation theorem, we proved using a weak convergence approach [1], the following large deviation result [5].

Theorem 2.4. (Large deviation principle) *For absolutely continuous functions φ from $\mathcal{D}([0,T],\mathbb{R}^d)$, with $T > 0$ arbitrary fixed, satisfying $\varphi(0) = u$, and for each fixed $k \in \hat{E}$, define*

$$I^{u,k}_{[0,T]}(\varphi) := \int_0^T L(\varphi(t),\dot{\varphi}(t);k)dt, \tag{2.6}$$

where L is subsequently defined. For all other functions in $\mathcal{D}([0,T],\mathbb{R}^d)$, $I^{u,k}_{[0,T]}(\varphi) := \infty$. Then the family $\xi^{\varepsilon}(t), \varepsilon > 0$ satisfies the large deviation principle with rate function

$$I^{u}_{[0,T]}(\varphi) = \min\{I^{u,k}_{[0,T]}(\varphi) : 1 \leq k \leq N\} \tag{2.7}$$

Let $\mathbb{L}^{\epsilon}$ be the infinitesimal generator of the family of coupled Markov processes $(\xi^{\epsilon}(t), x^{\epsilon}(\frac{t}{\epsilon}))$, $t \geq 0$, $\epsilon > 0$ on $\mathbb{R}^d \times E$ defined in (2.2).

Consider the martingale problem for the generator $\mathbb{L}^{\epsilon}$ and its relationship with the exponential martingale problem by taking the transformation H^{ϵ} defined as

$$H^{\epsilon}f := \epsilon e^{-\frac{1}{\epsilon}f}\mathbb{L}^{\epsilon}e^{\frac{1}{\epsilon}f}$$

An important step is to prove the convergence of H^{ε} for an appropriate collection of sequences f^{ε} to an operator H in the sense that if f^ε converges to f as $\varepsilon \to 0$ the $H^\varepsilon f^\varepsilon$ converges to Hf. The limit operator is

$$\hat{H}f(u;k) = \hat{a}(u;k)f'(u) + \int_{\mathbb{R}^d} (e^{vf'(u)} - 1 - vf'(u))\hat{\Gamma}(u, dv; k)$$

We further associate to it the function in u and p in $\mathbb{R}^d$ defined by

$$H(u,p;k) := \hat{a}(u;k)p + \int_{\mathbb{R}^d} (e^{vp} - 1 - vp)\hat{\Gamma}(u, dv; k)$$

As shown in [5] the Legendre-Fenchel transform of H defined as

$$L(u,q;k) := \sup_{p\in\mathbb{R}^d} \{pq - H(u,p;k)\}$$

leads to the rate function

$$I^{u,k}_{[0,T]}(\varphi) := \int_0^T L(\varphi(t), \dot{\varphi}(t); k)dt.$$

The stochastic additive functional $\xi^\varepsilon(t)$ satisfies LDP i.e.: for any set $A \in \mathcal{D}([0,T], \mathbb{R}^d)$ and any function φ such that $\varphi(0) = u$, and for every point $u \in \mathbb{R}^d$ and $\varepsilon > 0$ there will correspond the probability measure $\mathbb{P}^\varepsilon_u$ such that

$$\begin{aligned} -\inf_{\varphi\in A^\circ} I^u_{[0,T]}(\varphi) &\leq \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^\varepsilon_u\{\xi^\varepsilon_u \in A\} \\ &\leq \limsup \varepsilon \log \mathbb{P}^\varepsilon_u\{\xi^\varepsilon_u \in A\} \leq -\inf_{\varphi\in\bar{A}} I^u_{[0,T]}(\varphi) \end{aligned}$$

where A° and $\bar{A}$ represent the interior respectively the closure of the set Γ.

Let ρ be Skorohod metric in $\mathcal{D}([0,T], \mathbb{R}^d)$, and let $A = \{\varphi \in \mathcal{D}([0,T], \mathbb{R}^d) : \rho(\varphi, \hat{\xi}_u) > \delta\ \}$ for any $\delta > 0$. Then

$$\begin{aligned} -\inf_{\varphi\in A^\circ} I^u_{[0,T]}(\varphi) &\leq \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^\varepsilon_u\{\rho(\xi^\varepsilon_u, \hat{\xi}_u) > \delta\} \\ &\leq \limsup \varepsilon \log \mathbb{P}^\varepsilon_u\{\rho(\xi^\varepsilon_u, \hat{\xi}_u) > \delta\} \leq -\inf_{\varphi\in\bar{A}} I^u_{[0,T]}(\varphi) \end{aligned}$$

Corollary 2.5. *$\mathbb{P}^\varepsilon_u\{\rho(\xi^\varepsilon_u, \hat{\xi}_u) > \delta\} \leq \exp(-c\frac{1}{\varepsilon})$, where $c \leq \inf_{\varphi\in\bar{A}} I^u_{[0,T]}(\varphi)$.*

Let $\Phi_u(b) = \{\varphi \in \mathcal{C}([0,T], \mathbb{R}^d) : I^u_{[0,T]}(\varphi) \leq b\}$ be a compact set in $\mathcal{C}([0,T], \mathbb{R}^d)$.

Corollary 2.6. *For any function φ such that $\varphi(0) = u$ and for any b, δ, β positive,*

$$\mathbb{P}_u^\varepsilon\{\rho(\xi^\varepsilon, \varphi) < \delta\} \geq \exp\{-\tfrac{1}{\varepsilon}(I_{[0,T]}^u(\varphi) + \beta)\}$$
$$\mathbb{P}_u^\varepsilon\{\rho(\xi^\varepsilon, \Phi_u(b)) > \delta\} \leq \exp\{-\tfrac{1}{\varepsilon}(b - \beta)\}$$

3. Exit-time problem for ergodic switching Markov process

Assume that the jump Markov process is ergodic on E and the average system defined by

$$\frac{d}{dt}\hat{\xi}(t) = \hat{a}(\hat{\xi}(t); \hat{x}(t)), \qquad \hat{\xi}(0) = \xi(0) \tag{3.1}$$

where the averaged velocity is determined by

$$\hat{a}(u) = \int_E \pi(dx) a(u; x)$$

has an asymptotically stable position ϑ. Let $D \in \mathbb{R}^d$ be a bounded domain attracted to the asymptotically stable position ϑ, i.e. $\hat{a}(u) = 0$. The stochastic additive functional switched by the ergodic jump Markov process satisfies large deviation principle with the rate function given in (2.6) with $k = 1$, that is

$$I_{[0,T]}^u = \int_0^T L(\varphi(t), \dot{\varphi}(t))dt. \tag{3.2}$$

Define by $\mathcal{E}_\alpha(u)$ the α-neighborhood of a point $u \in \mathbb{R}^d$, for any $\alpha > 0$.

Proposition 3.1. *Assume that D is attracted to the equilibrium position ϑ and for $u \in \partial D$, $(\hat{a}(u), n(u)) < 0$ where $n(u)$ is the interior normal. Then for any $\alpha > 0$ we have:*

- (i) *there exist positive constants c_1 and T_0 such that for any function φ_t assuming its values in the set $D \cup \partial D - \mathcal{E}_\alpha(\vartheta)$ for $t \in [0, T]$, we have $I_u(\varphi) > c_1(T - T_0)$, $T > T_0$.*
- (ii) *there exist positive constants c_2 and T_0 such that for all sufficiently small $\varepsilon > 0$ and any $u \in D \cup \partial D - \mathcal{E}_\alpha(\vartheta)$ we have*

$$\mathbb{P}_u\{\zeta_\alpha > T\} \leq \exp\{-\frac{1}{\varepsilon}c_2(T - T_0)\} \tag{3.3}$$

where $\zeta_\alpha := \inf\{t > 0 : \xi_t^\varepsilon \notin D - \mathcal{E}_\alpha(\vartheta))\}$.

Proof. The proof follows along the lines of Freidlin-Wentzell [2]. Part (i) is similar to Lemma 2.2 in [2] and for the sake of completeness we provide the details adapted to our situation. In part (ii) we have to take into the account that ξ_t^ε is not a Markov process as in Lemma 2.2 cited above. We use the fact that the coupled process $(\xi_t^\varepsilon, \hat{x}(\frac{t}{\varepsilon}))$ is a Markov process to obtain similar property.

(i) Since the domain D is attracted to ϑ there exists $\mathcal{E}_{\alpha'}(\vartheta) \subset \mathcal{E}_\alpha(\vartheta)$ such that the trajectories of $\xi_t^\varepsilon(u)$ starting in $\mathcal{E}_{\alpha'}(\vartheta)$ never leave $\mathcal{E}_\alpha(\vartheta)$. Let $T(\alpha', u)$ be the time spent by $\hat{\xi}_t(u)$ until reaching $\mathcal{E}_{\alpha'}(\vartheta)$. It follows that $T(\alpha', u) < \infty$ for any $u \in D \cup \partial D$ and $T(\alpha', u)$ is upper semicontinuous in u, therefore it attains its largest value $T_0 = \max_{u \in D \cup \partial D} T(\alpha', u) < \infty$.

The set of functions from $\mathcal{C}([0, T_0], D \cup \partial D - \mathcal{E}_\alpha(\vartheta))$ is closed in $\mathcal{C}([0, T])$, so $I^u_{[0,T_0]}(\varphi)$ attains its infimum on this set. This infimum is different from zero since otherwise some trajectory of the dynamical system would belong to this set. Therefore $I^u_{[0,T_0]} \geq A > 0$ and for any $T > T_0$, $I^u_{[0,T]} \geq A$ for any $\varphi \in \mathcal{C}([0, T], D \cup \partial D - \mathcal{E}_\alpha(\vartheta))$. Using the subadditivity of the rate function we get for $T > 2T_0$ that $I^u_{[0,T]}(\varphi) \geq 2A$ and so on. We get

$$I^u_{[0,T]}(\varphi) \geq A[\frac{T}{T_0}] > A(\frac{T}{T_0 - 1}) = a(T - T_0)$$

(ii) Since D is attracted to ϑ and $(\hat{a}(u), n(u)) < 0$ on the boundary of D it follows that the same properties will be enjoyed by the δ-neighborhoods of D for sufficiently small $\delta > 0$. Assume that $\delta < \frac{\alpha}{2}$. From (i) we get $I^u_{[0,T_0]}(\varphi) > 0$ for functions φ that do not leave the closed δ-neighborhood of D and do not get into $\mathcal{E}_{\frac{\alpha}{2}}(\vartheta)$. For $u \in D$ the functions in $\phi_u(A) = \{\varphi : \varphi(0) = u,\ I^u_{[0,T_0]}(\varphi) \leq A\}$ reach $\mathcal{E}_{\frac{\alpha}{2}}(\vartheta)$ or leave the δ-neighborhood of D during the time from 0 to T_0. The trajectories of $\xi_t^\varepsilon(u)$ for which $\zeta_\alpha(u) > T_0$ are at a distance not smaller than δ from this set, hence Corollary 2.6 yields

$$\mathbb{P}_u\{\zeta_\alpha > T_0\} \leq \exp\{-\frac{1}{\varepsilon}(A - \beta)\}.$$

Let us denote by $X_t^\varepsilon := (\xi_t^\varepsilon, x^\varepsilon(\frac{t}{\varepsilon}))$ the coupled Markov process. Let $\mathcal{F}_t$ be the σ-algebra generated by the trajectories $\mathcal{F}_t = \sigma(X_s^\varepsilon; s \leq t)$.

Then using the Markov property we have

$$\begin{aligned}
\mathbb{P}_u(\zeta^\alpha > (n+1)T_0) &= \mathbb{P}_u(\zeta^\alpha \circ \theta_{nT_0} > T_0; \zeta^\alpha > nT_0) \\
&= \mathbb{E}_u(1\!\!1_{\zeta^\alpha \circ \theta_{nT_0}} > T_0; \zeta^\alpha > nT_0) \\
&= \mathbb{E}_u(\mathbb{E}_u(1\!\!1_{\zeta^\alpha \circ \theta_{nT_0}} > T_0) | \mathcal{F}_{nT_0}); \zeta^\alpha > nT_0) \\
&= \mathbb{E}_u(\mathbb{P}_{X^\varepsilon_{nT_0}}(\zeta^\alpha > T_0); \zeta^\alpha > nT_0) \\
&= \mathbb{E}_u(\mathbb{P}_{\xi^\varepsilon_{nT_0}}(\zeta^\alpha > T_0); \zeta^\alpha > nT_0) \\
&\leq \mathbb{P}_u(\zeta^\alpha > nT_0) \sup_{y \in D} \mathbb{P}_y(\zeta^\alpha > T_0).
\end{aligned}$$

We get

$$\begin{aligned}
&\mathbb{P}_u(\zeta^\alpha > T) \leq \mathbb{P}_u(\zeta^\alpha > [\frac{T}{T_0}]T_0) \leq [\sup_{y \in D} \mathbb{P}_y(\zeta^\alpha > T_0)]^{[\frac{T}{T_0}]} \\
&\leq \exp\{-\frac{1}{\varepsilon}(A-\beta)(\frac{T}{T_0}-1)\} = \exp\{-\frac{1}{\varepsilon}c(T-T_0)\}.
\end{aligned}$$

□

Let us define the *quasipotential* with respect to $u \in \mathbb{R}^d$

(3.4)
$$V(u,v) = \inf\{I^u_{[0,T]}(\varphi) : \varphi(0) = u, \varphi(T) = v, \varphi(t) \in D \cup \partial D, \forall t \in [0,T]\}$$

This describe the difficulty of passage from the initial state u to a small neighborhood of v, without leaving $D \cup \partial D$ within a "reasonable" time, since

$$V(u,v) = \lim_{T\to\infty} \lim_{\delta\to 0} \lim_{\varepsilon\to 0}[-\varepsilon \log \mathbb{P}_u(\tau_\delta \leq T)] \tag{3.5}$$

where τ_δ be the first entrance time in the δ-neighborhood of v for the process ξ^ε_t, see for details [2].

In what follows we assume: $V(\vartheta, \partial D) < \infty$. Let $v_0 \in \partial D$ such that $V(\vartheta, v_0) = \min_{v \in \partial D} V(\vartheta, v)$.

Proposition 3.2. *The following properties hold:*

(i) $V(\vartheta, v) \geq 0$
(ii) $V(\vartheta, \vartheta) = 0$
(iii) $V(\vartheta, v)$ *is continuous in* v.

Theorem 3.3. *Let D be a bounded domain attracted to the equilibrium position ϑ. Suppose that there exists a unique point*

$v_0 \in \partial D$ such that $V(\vartheta, v_0) = \min_{v\in\partial D} V(\vartheta, v)$. Then for every $\delta > 0$ and $u \in D$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_u\{\rho(\xi^\varepsilon_{\tau^\varepsilon}, v_0) < \delta\} = 1$$

where $\tau^\varepsilon := \inf\{t > 0 : \xi^\varepsilon_t \in \partial D\}$.

Proof. We study the trajectories of ξ^ε_t starting from a small neighborhood of the equilibrium position ϑ. Let Γ and γ be small spheres of radii μ and $\mu/2$ with center in ϑ.

The process starting at $u \in \gamma$ will follow the trajectories given by the differential equation inside γ and stay inside Γ for a long time. The process may return to γ many times and with a very small probability the process may not return and exits from D. If the process exits from D then this happen with probability one around the point of minimum potential. To prove this we need to look at the returning times in $\gamma \cup \partial D$. Let's define the increasing sequence of returning times τ_0, σ_0, τ_1, $\sigma_1, \ldots$ such that: $\tau_0 = 0$, $\sigma_n = \inf\{t > \tau_n : \xi^\varepsilon_t \in \partial\Gamma\}$, $\tau_n = \inf\{t > \sigma_{n-1} : \xi^\varepsilon_t \in \gamma \cup \partial D\}$, with $\inf \emptyset = \infty$. Let's define a new process on the set $\gamma \cup \partial D$, $\eta^\varepsilon_n := \xi^\varepsilon_{\tau_n}$ and on E, $y^\varepsilon_n := x^\varepsilon(\frac{\tau^\varepsilon_n}{\varepsilon})$. The couple $X_n = (\eta^\varepsilon_n, y^\varepsilon_n)$ is a Markov chain.

First, let's suppose that $u \in \gamma$ and let N be the time of exit from D. Let $\mathbb{P}_{n,\varepsilon}$ be the probability of exiting from $\partial D - \mathcal{E}_\delta(v_0)$ during the n'th trip from Γ to $\gamma \cup \partial D$, where $\mathcal{E}_\delta(v_0)$ is the sphere with radius δ and center v_0.

$$\mathbb{P}_u\{|\xi^\varepsilon_{\tau^\varepsilon} - v_0| \geq \delta\} = \sum_{n=1}^{\infty} \mathbb{P}_{n,\varepsilon} = \sum_{n=1}^{\infty} \mathbb{P}_u\{\eta^\varepsilon_n \in \partial D - \mathcal{E}_\delta(v_0)\}$$

$$= \sum_{n=1}^{\infty} \mathbb{E}_u\{\mathbb{1}_{\{\eta^\varepsilon_1\circ\theta_{n-1}\in\partial D-\mathcal{E}_\delta(v_0)\}}; \eta^\varepsilon_1 \in \gamma, \ldots, \eta^\varepsilon_{n-1} \in \gamma\}$$

$$\sum_{n=1}^{\infty} \mathbb{E}_u\{\mathbb{E}_u\{\mathbb{1}_{\{\eta^\varepsilon_1\circ\theta_{n-1}\in\partial D-\mathcal{E}_\delta(v_0)\}}|\mathcal{F}_{n-1}\}\}\eta^\varepsilon_1 \in \gamma, \ldots, \eta^\varepsilon_{n-1} \in \gamma\}$$

$$\sum_{n=1}^{\infty} \mathbb{E}_u\{\mathbb{E}_{X_{n-1}}\{\eta^\varepsilon_1 \in \partial D - \mathcal{E}_\delta(v_0)\}; \eta^\varepsilon_1 \in \gamma, \ldots, \eta^\varepsilon_{n-1} \in \gamma\}$$

$$\sum_{n=1}^{\infty} \mathbb{E}_u\{\mathbb{E}_{\eta_{n-1}}\{\eta^\varepsilon_1 \in \partial D - \mathcal{E}_\delta(v_0)\}; \eta^\varepsilon_1 \in \gamma, \ldots, \eta^\varepsilon_{n-1} \in \gamma\}$$

$$\leq \sum_{n=1}^{\infty} \mathbb{E}_u\{\eta_1^\varepsilon \in \gamma, ..., \eta_{n-1}^\varepsilon \in \gamma\} \mathbb{P}_{\eta_{n-1}}\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\}$$

$$\leq \sum_{n=1}^{\infty} \mathbb{P}_u(N = n) \mathbb{P}_{\eta_{n-1}}\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\}$$

Let's assume that we have proved for $u \in \gamma$, $\mathbb{P}_u\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\} \to 0$. Then the above sum converges to zero and so the theorem is proved for $u \in \gamma$.
Let's consider the general case that $u \in D$. We write

$$\begin{aligned}\mathbb{P}_u(|\xi^\varepsilon_{\tau^\varepsilon} - v_0| \geq \delta) &\leq \mathbb{P}_u(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \in \partial D) + \mathbb{P}_u(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \in \gamma, |\xi^\varepsilon_{\tau^\varepsilon} - v_0| \geq \delta) \\ &= \mathbb{P}_u(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \in \partial D) + \mathbb{E}_u(\mathbb{P}_{\xi^\varepsilon_{\tau(\gamma\cup\partial D)}}(|\xi^\varepsilon_{\tau^\varepsilon} - v_0| \geq \delta); \xi^\varepsilon_{\tau(\gamma\cup\partial D)} \in \gamma).\end{aligned}$$

and both terms converge to zero.
It only remains to prove that for $u \in \gamma$, $\mathbb{P}_u\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\} \to 0$.

As in [2], lemma 2.4, it can be shown that for $u \in \gamma$,

$$\mathbb{P}_u\{\eta_1^\varepsilon \in \partial D\} \geq \exp\{-\frac{1}{\varepsilon}(V(\vartheta, v_0) + 0.45d)\} \tag{3.6}$$

where $d = \min\{V(\vartheta, u) : v \in \partial D, |v - v_0| \geq \delta\} - V(\vartheta, v_0)$. We will prove now that for $u \in \gamma$,

$$\mathbb{P}_u\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\} \leq \exp\{-\frac{1}{\varepsilon}(V(\vartheta, v_0) + 0.55d)\} \tag{3.7}$$

Let $\tau(\gamma \cup \partial D) := \inf\{t > 0 : \xi_t^\varepsilon \in \gamma \cup \partial D\}$. So η_1^ε is $\xi^\varepsilon_{\tau(\gamma\cup\partial D)}$ shifted by σ_0. Using Markov property we get

$$\begin{aligned}&\mathbb{P}_u\{\eta_1^\varepsilon \in \partial D - \mathcal{E}_\delta(v_0)\} = \mathbb{P}_u\{\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \circ \theta_{\sigma_0} \in \partial D - \mathcal{E}_\delta(v_0)\} \\ &= \mathbb{E}_u(\mathbb{E}_u(\mathbb{1}_{\{\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \circ \theta_{\sigma_0} \in \partial D - \mathcal{E}_\delta(v_0)\}} | \mathcal{F}_{\sigma_0}) \\ &= \mathbb{E}_u(\mathbb{P}_{X^\varepsilon_{\sigma_0}}(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \circ \theta_{\sigma_0} \in \partial D - \mathcal{E}_\delta(v_0)) \\ &= \mathbb{E}_u(\mathbb{P}_{\xi^\varepsilon_{\sigma_0}}(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \circ \theta_{\sigma_0} \in \partial D - \mathcal{E}_\delta(v_0))) \\ &\leq \sup_{v\in\Gamma} \mathbb{P}_v(\xi^\varepsilon_{\tau(\gamma\cup\partial D)} \circ \theta_{\sigma_0} \in \partial D - \mathcal{E}_\delta(v_0)).\end{aligned}$$

From Proposition 3.1 (ii), there exists $T > 0$ such that

$$\mathbb{P}_u(\tau(\gamma \cup \partial D) > T) \leq \exp\{-\frac{1}{\varepsilon}(V(\vartheta, v_0) + d)\}. \tag{3.8}$$

Let K be the closure of the $\frac{\mu}{2}$-neighborhood of $\partial D - \mathcal{E}_\delta(v_0)$. Since no function φ_t, $0 \leq t \leq T$, $\varphi_0 \in \Gamma$ such that $I^u_{[0,T]}(\varphi) \leq V(\vartheta, v_0) + 0.65d$

hits K, $\cup_{u\in\Gamma}\phi_u(V(\vartheta,v_0)+0.65d)$ pass at a distance not smaller than $\frac{\mu}{2}$ from $\partial D-\mathcal{E}_\delta(v_0)$. Using Corollary 2.6 we get

$$\mathbb{P}_u\{\tau(\gamma\cup\partial D)\leq T,\, \xi^\varepsilon_{\tau(\gamma\cup\partial D)}\in\partial D-\mathcal{E}_\delta(v_0)\}$$
$$\leq \mathbb{P}_u\{\rho_{0T}(\xi^\varepsilon,\phi_u(V(\vartheta,v_0)+0.65d)\geq\frac{\mu}{2}\}$$
$$\leq \exp\{-\frac{1}{\varepsilon}(V(\vartheta,v_0)+0.65d-0.05d)=\exp\{-\frac{1}{\varepsilon}(V(\vartheta,v_0)+0.60d).$$

Also

$$\mathbb{P}_u\{\xi^\varepsilon_{\tau(\gamma\cup\partial D)}\in\partial D-\mathcal{E}_\delta(v_0)\}\leq\mathbb{P}_u\{\tau(\gamma\cup\partial D)>T\}+$$
$$\mathbb{P}_u\{\tau(\gamma\cup\partial D)\leq T,\, \xi^\varepsilon_{\tau(\gamma\cup\partial D)}\in\partial D-\mathcal{E}_\delta(v_0)\}$$
$$\leq\exp\{-\frac{1}{\varepsilon}(V(\vartheta,v_0)+d)\}+\exp\{-\frac{1}{\varepsilon}(V(\vartheta,v_0)+0.60d)\}$$
$$=\exp\{-\frac{1}{\varepsilon}(V(\vartheta,v_0)+0.55d)\}.$$

□

Theorem 3.4. *Let $v_0\in\partial D$ for which $V(\vartheta,v_0)=min_{v\in\partial D}V(\vartheta,v)$. Then the solution of the Dirichlet problem* (2.4) *has the property*

$$\lim_{\varepsilon\to0}\varphi_\varepsilon(u;x)=f(v_0;x),\quad u\in D,\, x\in E$$

Proof.

$$\begin{aligned}\lim_{\varepsilon\to0}\varphi_\varepsilon(u;x) &= \lim_{\varepsilon\to0}\mathbb{E}_{\mathbf{P}^\varepsilon_{u;x}}(f(\xi^\varepsilon(\tau^\varepsilon;x)\\ &= \mathbb{E}_{\mathbf{P}_{u;x}}[\lim_{\varepsilon\to0}f(\xi^\varepsilon(\tau^\varepsilon;x)]\\ &= \lim_{\varepsilon\to0}f(\xi^\varepsilon(\tau^\varepsilon;x)=f(v_0,x).\end{aligned}$$

□

4. Exit time problem for additive functionals in split spaces

Assume that the limit process $\hat{\xi}(t),t\geq0$ defined in (2.3) has a unique asymptotically stable position. For a fixed k suppose that ϑ_k is the asymptotically stable position of the deterministic system

$$\frac{d}{dt}\hat{\xi}(t)=\hat{a}(\hat{\xi}(t);k)$$

where

$$\hat{a}(u;k)=\int_{E_k}\pi_k(dx)a(u,x).$$

The stochastic additive functional switched by the jump Markov process in split space satisfies LDP with rate function

$$I^u_{[0,T]}(\varphi) = \min\{I^{u,k}_{[0,T]}(\varphi) : 1 \leq k \leq N\} \tag{4.1}$$

where $I^{u,k}_{[0,T]}(\varphi)$ is defined in (2.6). We associate to it the corresponding quasipotential as in (3.4).
Following the same procedure as in Theorem 3.3 with the rate function replaced by 4.1, we get the following result.

Theorem 4.1. *Let $D \in \mathbb{R}^d$ be a bounded domain attracted to the equilibrium position ϑ_k. Suppose that there exists a unique point $v_k \in \partial D$ such that $V(\vartheta_k, v_k) = \min_{v\in\partial D} V(\vartheta, v)$. Then for every $\delta > 0$ and $u \in D$,*

$$\lim_{\varepsilon\to 0} \mathbb{P}_u\{\rho(\xi^\varepsilon_{\tau^\varepsilon}, v_k) < \delta\} = 1$$

where $\tau^\varepsilon := \inf\{t > 0 : \xi^\varepsilon_t \in \partial D\}$.

References

[1] P. Dupuis and R.S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. John Wiley & Sons, New York, 1997.

[2] M. I. Freidlin and A.D. Wentzel. *Random Perturbations of Dynamical Systems*. Springer-Verlag, New York, 1998.

[3] V.S. Koroliuk and N. Limnios. *Stochastic Systems in Merging Phase Space*. World Scientific, London, 2005.

[4] Bernt Oksendal. *Stochastic Differential Equations*. Springer, Berlin Heidelberg, 1998.

[5] A. Oprisan and A. Korzeniowski. Large deviations for ergodic processes in split spaces. *Dynamic Systems and Applications*, 18, 2009.

[6] Adam Shwartz and Alan Weiss. *Large Deviations for Performance Analysis// Queues, communications, and computing*. Chapman & Hall, UK, 1995.

[7] Anatoly Swishchuk. *Random Evolutions and Their Applications*. Kluwer Academic Publishers, Netherlands, 2000.