LARGE DEVIATIONS FOR ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

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Abstract: We prove a large deviation principle for a class of empirical processes in $\mathcal{C}[0,\infty)$ associated with additive functionals of Markov processes that were shown to have a martingale decomposition representation.

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1. Introduction

We study a class of empirical measures on $\mathcal{C}[0,\infty)$ associated with $S_n(g) = \sum_{k=0}^{n-1} g(X_k)$, where (X_k) is a Markov chain with an invariant measure m and $g \in L^2(m)$, via an invariance principle corresponding to interpolation of S_n.

Section two introduces the necessary background and provides auxiliary results for subsequent analysis. In section three we provide criteria which allow us to establish a martingale decomposition

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representation for $S_n(g)$, the key element in the proof of the large deviation result presented in the last section of the paper.

2. Auxiliary results

2.1. Functional almost everywhere central limit theorems for additive functionals. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(X_i) = 0$, $\mathbb{E}(X_i^2) = 1$ and $S_n = \sum_{i=1}^n X_i$. Define the interpolation processes, with $\psi_n\left(\frac{k}{n}\right) = S_k$ for $1 \le k \le n$,

$$\psi_n(t) := \frac{1}{\sqrt{n}}\left(S_{[nt]} + (nt - [nt])(S_{[nt]+1} - S_{[nt]})\right), \; 0 \le t \le 1 \tag{2.1}$$

and empirical processes $W_n : \mathcal{C}[0,1] \to \mathcal{M}_1(\mathcal{C}[0,1])$,

$$W_n(\cdot) := \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{\{\psi_k \in \cdot\}} \tag{2.2}$$

with $L(n) = \sum_{k=1}^n \frac{1}{k}$. The following functional almost everywhere central limit theorem is due to Brosamler [**1**].

Theorem 2.1. *The random process W_n converges weakly to the Wiener measure W on $\mathcal{C}[0,1]$, $\mathbb{P}$-a.e.*

Let (X_n) be an ergodic Markov chain with stationary measure m, $S_n(g) = \sum_{k=0}^{n-1} g(X_k)$ be such that $\mathbb{E}_m(S_1^2) = \sigma^2 \in (0, \infty)$ and for any initial distribution μ and any $k \in \mathbb{N}$, $\mathbb{E}_\mu(S_k^2) < \infty$. Define the interpolation processes

$$\Psi_n(t) := \frac{1}{\sigma\sqrt{n}}\left(S_{[nt]} + (nt - [nt])(S_{[nt]+1} - S_{[nt]})\right), \; 0 \le t < \infty \tag{2.3}$$

and the empirical processes

$$W_n(\cdot) := \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{\{\Psi_k \in \cdot\}} \tag{2.4}$$

Theorem 2.2. *W_n converges weakly to the Wiener measure W on $\mathcal{C}[0,\infty)$, $\mathbb{P}_\mu$-a.e.*

Proof. By using our martingale decomposition (see section 3, Theorem 3.1) we can write $S_n = M_n + R_n$, where (M_n) is a mean zero martingale for which $\sup_{1 \le k \le n} \frac{|R_k|}{\sigma\sqrt{n}}$ converges in probability to

0. Define the empirical measures

$$W_n^M(\cdot) = \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{\{\Psi_k^M \in \cdot\}} \tag{2.5}$$

where Ψ_n^M is the interpolation process

$$\Psi_n^M(t) = \frac{1}{\sigma\sqrt{n}} \{M_{[nt]} + (nt - [nt])(M_{[nt]+1} - M_{[nt]}) \tag{2.6}$$

corresponding to the martingale $M = (M_n)$. By [**8**], W_n^M converges weakly to the Wiener measure W on $\mathcal{C}[0,1]$, i.e., for any bounded continuous function $f : \mathcal{C}[0,1] \to \mathbb{R}$, W- a.e.,

$$\lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{f\circ\Psi_k^M} = \phi(W)$$

To show that W_n converges weakly to the Wiener measure W on $\mathcal{C}[0,1]$, we check that

$$\lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{f\circ\Psi_k} = \lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{f\circ\Psi_k^M}$$

for which it is sufficient to have $\lim_{n\to\infty} ||\Psi_n - \Psi_n^M||_{\mathcal{C}[0,1]} = 0$. Indeed,

$$||\Psi_n - \Psi_n^M||_{\mathcal{C}[0,1]} = \sup_{t\in[0,1]} |\Psi_n(t) - \Psi_n^M(t)| =$$

$$\sup_{t\in[0,1]} \left| \frac{1}{\sigma\sqrt{n}} \{R_{[nt]} + (nt - [nt])(R_{[nt]+1} - R_{[nt]})\} \right| \leq$$

$$\sup_{1\leq k\leq n} \sup_{t\in\left[\frac{k-1}{n}, \frac{k}{n}\right]} \frac{1}{\sigma\sqrt{n}} |R_{[nt]} + (nt - [nt])(R_{[nt]+1} - R_{[nt]})| \leq$$

$$\sup_{1\leq k\leq n} \frac{1}{\sigma\sqrt{n}} |R_k|$$

which by (3.2) converges to 0 in probability.

By replacing 1 with N in the above proof the result holds for $\mathcal{C}[0,N]$, for every $N > 0$. The extension to $\mathcal{C}[0,\infty)$ is standard: ν_n converges weakly to ν on $\mathcal{C}[0,\infty)$ if for every $N > 0$, $\pi_N \nu_n$ converges weakly to $\pi_N \nu$ on $\mathcal{C}[0,N]$, where π_N of a measure on $\mathcal{C}[0,\infty)$ denotes the measure it induces on $\mathcal{C}[0,N]$.

In our case this means that W_n converges weakly to the Wiener measure W on $\mathcal{C}[0,\infty)$. □

2.2. Large deviation principle for Ornstein-Uhlenbeck processes. Define the canonical shifts $\sigma_t : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ by

$$\sigma_t\omega(s) := \omega(t+s)$$

and empirical processes $R_T : \mathcal{C}(\mathbb{R}) \to \mathcal{M}_1(\mathcal{C}(\mathbb{R}))$ for any $T > 0$ by

$$R_T := \frac{1}{2T}\int_{-T}^{T} \delta_{\sigma_t\omega} dt.$$

Assume that σ is an ergodic transformation, and let $\mathbb{P}$ be a σ-invariant measure i.e. $\mathbb{P}$ is the probability measure on the space of trajectories corresponding to the Ornstein-Uhlenbeck processes starting with stationary distribution at time $t = 0$.

Let $\Omega = \mathcal{C}(\mathbb{R})$, then for any bounded continuous function we have:

$$\begin{aligned}\int_\Omega f(\omega)dR_T(\omega) &= \frac{1}{2T}\int_{-T}^{T}\left(\int_\Omega f(\omega)d\delta_{\sigma_t\omega}\right)dt = \frac{1}{2T}\int_{-T}^{T} f(\sigma_t\omega)dt \\ &\to \int_\Omega f(\omega)d\mathbb{P}\end{aligned}$$

a.s. as $T \to \infty$ by ergodic theorem. This implies the weak convergence $R_T \Rightarrow \mathbb{P}$ as $T \to \infty$. The exponential decay for the deviations of R_T from $\mathbb{P}$ was given by Donsker and Varadhan in [**2**] through the rate function $H(Q)$, $Q \in \mathcal{M}_1(\mathcal{C}(\mathbb{R}))$ defined as

$$H(Q) := \begin{cases} \lim_{T\to\infty} \frac{1}{T} h(\Im|_{[-T,T]}(Q)/\Im|_{[-T,T]}(\mathbb{P})) & \text{if } Q \text{ is } \sigma \text{ invariant} \\ \infty & \text{otherwise}\end{cases}$$

where $\Im|_I(\mu)$ denote the image of a measure μ under the restriction map on I and $h(\mu/\nu)$ is the relative entropy of μ with respect to ν,

$$h(\mu/\nu) := \begin{cases} \int \log(\frac{d\mu}{d\nu}) & \text{if } \mu \ll \nu \\ \infty & \text{otherwise}\end{cases} \tag{2.7}$$

Theorem 2.3. *For every bounded interval $I \subset \mathbb{R}$,*

$$H_I(\cdot) := \inf_{Q\in\Im|_I^{-1}(\cdot)} H(Q)$$

is a rate function on $\mathcal{M}_1(\mathcal{C}(I))$ and for any Borel set $A \subseteq \mathcal{M}_1(\mathcal{C}(I))$,

$$\begin{aligned}-\inf_{A^\circ} H_I &\le \liminf_{T\to\infty} \frac{1}{T}\log \mathbb{P}\{\Im|_I(R_T) \in A\} \\ &\le \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{P}\{\Im|_I(R_T) \in A\} \le -\inf_{\bar{A}} H_I\end{aligned}$$

An extension to paths on $[0,\infty)$ by using a finer topology than the topology of uniform convergence on compact sets was obtained by Heck in [**4**].

Theorem 2.4. *Let $\Phi : \mathbb{R} \to [0,\infty)$ be a continuous function satisfying*

$$\lim_{t\to\infty} \frac{\Phi(t)}{\sqrt{t}} = \lim_{t\to-\infty} \frac{\Phi(t)}{\sqrt{|t|}} = \infty \tag{2.8}$$

and $\mathcal{C}_\Phi := \{\omega \in \mathcal{C}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \frac{|\omega(t)|}{\Phi(t)} < \infty\}$, $\mathcal{M}_1(\mathcal{C}_\Phi) = \{Q \in \mathcal{M}_1(\mathcal{C}(\mathbb{R})) : Q(\mathcal{C}_\Phi) = 1\}$. Then $H|_{\mathcal{M}_1(\mathcal{C}_\Phi)}$ is a rate function on $\mathcal{M}_1(\mathcal{C}_\Phi)$, and for every Borel set $A \subseteq \mathcal{M}_1(\mathcal{C}_\Phi)$,

$$\begin{aligned} -\inf_{A^\circ} H &\le \liminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}\{R_T \in A\} \\ &\le \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}\{R_T \in A\} \le -\inf_{\bar{A}} H \end{aligned}$$

Remark 2.5. Under condition (2.8), $\mathbb{P} \in \mathcal{M}_1(\mathcal{C}_\Phi)$ and $\mathbb{P}$-a.e. $R_T \in \mathcal{M}_1(\mathcal{C}_\Phi)$.

2.3. Large deviation principle for Brownian motion. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that

$$\lim_{t\to 0} \frac{\phi(t)}{\sqrt{t|\log t|}} = \lim_{t\to\infty} \frac{\phi(t)}{\sqrt{t|\log t|}} = \infty, \tag{2.9}$$

and the set $\mathcal{C}_\phi$ defined as

$$\mathcal{C}_\phi := \{\omega \in \mathcal{C}[0,\infty) : \sup_{t\in\mathbb{R}_+} \frac{|\omega(t)|}{\phi(t)} < \infty\}$$

Consider the isomorphism $F : \mathcal{C}(\mathbb{R}) \to \mathcal{C}[0,\infty)$, $F(\omega)(t) := \sqrt{t}\,\omega(\log t)$. Then $F|_{\mathcal{C}_\Phi} : \mathcal{C}_\Phi \to \mathcal{C}_\phi$ is a bijective isometry with respect to $|\cdot|_\Phi$ and $|\cdot|_\phi$ if $\Phi(\cdot) = \frac{\phi(\exp(\cdot))}{\sqrt{\exp(\cdot)}}$. Let us denote $W := \Im_F(\mathbb{P})$, so W is Wiener measure on $\mathcal{C}[0,\infty)$. Theorem 2.4 can be written in terms of Brownian motion as follows:
For any $t \in \mathbb{R}$ define $\theta_t : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ by $\theta_t\omega(s) := e^{-t/2}\omega(e^t s)$ and for any $T > 0$ empirical processes $S_T : \mathcal{C}[0,\infty) \to \mathcal{M}_1(\mathcal{C}[0,\infty))$ by

$$S_T(\omega) := \frac{1}{2T} \int_{-T}^{T} \delta_{\theta_t\omega} dt$$

Theorem 2.6. *Define for any Q in $\mathcal{M}_1(\mathcal{C}[0,\infty))$ the function*
(2.10)
$$I(Q) := \begin{cases} \lim_{T\to\infty} \frac{1}{T} h(\Im|_{[e^{-T},e^{T}]}(Q)/\Im|_{[e^{-T},e^{T}]}(W)) & \text{if } Q \text{ is } \theta\text{-invariant} \\ \infty & \text{otherwise} \end{cases}$$

Then $I|_{\mathcal{M}_1(\mathcal{C}_\phi)}$ is a rate function and for any Borel set $A \subseteq \mathcal{M}_1(\mathcal{C}_\phi)$,

$$\begin{aligned} -\inf_{A^\circ} I &\leq \liminf_{T\to\infty} \frac{1}{T} \log W\{S_T \in A\} \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \log W\{S_T \in A\} \leq -\inf_{\bar{A}} I \end{aligned}$$

where $\mathcal{M}_1(\mathcal{C}_\phi) := \{Q \in \mathcal{M}_1(\mathcal{C}[0,\infty)) : Q(\mathcal{C}_\phi) = 1\}$

Let's define for any $a > 0$, $\theta_a : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ by

$$\theta_a \omega(t) = \frac{1}{\sqrt{a}} \omega(at)$$

and empirical processes $\mathcal{R}_T : \mathcal{C}[0,\infty) \to \mathcal{M}_1(\mathcal{C}[0,\infty))$,

$$\mathcal{R}_T := \frac{1}{\log T} \int_{\frac{1}{\sqrt{T}}}^{\sqrt{T}} \delta_{\theta_t \omega} \frac{dt}{t}$$

Then $(\mathcal{R}_n)$ satisfies LDP with constants $(\log n)$ and rate function I, where
(2.11)
$$I(Q) := \begin{cases} \lim_{a\to\infty} \frac{1}{2\log a} h\left(Q \circ |^{-1}_{[\frac{1}{a},a]} / W \circ |^{-1}_{[\frac{1}{a},a]} \right) & \text{if } Q \text{ is } \theta\text{-invariant} \\ \infty & \text{otherwise} \end{cases}$$

which means

$$\begin{aligned} -\inf_{A^\circ} I &\leq \liminf_{n\to\infty} \frac{1}{\log n} \log W\{\mathcal{R}_n \in A\} \\ &\leq \limsup_{n\to\infty} \frac{1}{\log n} \log W\{\mathcal{R}_n \in A\} \leq -\inf_{\bar{A}} I \end{aligned}$$

Since $\frac{1}{\log n} \int_1^n \delta_{\theta_t} \frac{dt}{t}$ has under W the same distribution as $\frac{1}{\log n} \int_{\frac{1}{\sqrt{n}}}^{\sqrt{n}} \delta_{\theta_t} \frac{dt}{t}$ we have the following corollary:

Corollary 2.7. *Let $\widetilde{\mathcal{R}}_n = \frac{1}{\log n} \int_1^n \delta_{\theta_t} \frac{dt}{t}$. Then $(\widetilde{\mathcal{R}}_n)$ satisfies LDP with constants $(\log n)$ and rate function I defined in (2.11).*

2.4. Large deviation principle for sequence of i.i.d. random variables. Let's consider $X_1, X_2, ...$ i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(X_i) = 0$, $\mathbb{E}(X_i^2) = 1$ and $S_n = \sum_{i=1}^n X_i$. We want to prove the LDP for the functional almost everywhere central limit theorem given in Theorem 2.1.

Lemma 2.8. *Let Y_n and Z_n be random variables with values in a metric space (E, d) such that for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d(Y_n, Z_n) > \epsilon\} = -\infty \tag{2.12}$$

Then Y_n and Z_n are equivalent with respect to LDP (or exponentially equivalent), which means that if (Y_n) satisfies LDP with constants $(\log n)$ and rate function I, then (Z_n) also satisfies LDP with the same constants and rate function.

To use this lemma for sequences of random variables in $\mathcal{M}_1(\mathcal{C}_\phi)$ it is necessary to define a metric, as done in [**4**].

Let $(\mathcal{C}_\phi, |\cdot|_\phi)$ be a metric space, with $|\omega|_\phi = \sup_{t\in\mathbf{R}_+} \frac{|\omega(t)|}{\phi(t)}$. On $\mathcal{M}_1(\mathcal{C}_\phi)$ define

$$d_\phi(\mu,\nu) := \sup\left\{\left|\int f\,d\mu - \int f\,d\nu\right|, f\in\mathcal{C}(\mathcal{C}_\psi, \mathbb{R}), \|f\|_L \le \frac{1}{2}\right\} \tag{2.13}$$

where $\|f\|_L := \sup_{\omega\in\mathcal{C}_\phi} |f(\omega)| + \sup_{\omega,\omega'\in\mathcal{C}_\phi,\omega\neq\omega'} \frac{|f(\omega)-f(\omega')|}{|\omega-\omega'|_\phi}$. Then $(\mathcal{M}_1(\mathcal{C}_\phi), d_\phi)$ becomes a metric space and the following properties hold:

(a) $d_\phi \le 1$
(b) $d_\phi(\alpha\mu+(1-\alpha)\nu, \alpha\tilde{\mu}+(1-\alpha)\tilde{\nu}) \le \alpha d_\phi(\mu,\tilde{\mu})+(1-\alpha)d_\phi(\nu,\tilde{\nu})$, for $\alpha\in[0,1]$ and $\mu,\tilde{\mu},\nu,\tilde{\nu}\in\mathcal{M}_1(\mathcal{C}_\phi)$
(c) $d_\phi(\delta_\omega, \delta_{\omega'}) \le |\omega-\omega'|_\psi$ for $\omega,\omega'\in\mathcal{C}_\phi$.

Then, as shown in [**3**], we have

Lemma 2.9. *Let Y_t,Z_t, $t\in[1,\infty)$ be random variables with values in the separable metric space $(\mathcal{C}_\phi, |\cdot|_\phi)$ such that $t\to Y_t$, $t\to Z_t$ are continuous from the right or left and for all $\varepsilon > 0$,*

$$\lim_{l\to\infty} \frac{1}{\log l} \log \mathbb{P}\left\{\sup_{s\in[l,l+1)} |Y_s - Z_s|_\phi > \varepsilon\right\} = -\infty \tag{2.14}$$

Then $\left(\frac{1}{\log n}\int_1^n \delta_{Y_s}\frac{ds}{s}\right)$ *and* $\left(\frac{1}{\log n}\int_1^n \delta_{Z_s}\frac{ds}{s}\right)$ *are equivalent with respect to LDP.*

Let $W_n = \frac{1}{L(n)}\sum_{k=1}^n \frac{1}{k}\delta_{\psi_k}$ be as defined in (2.2) and ϕ above satisfies in addition $\phi(t) > t^{\frac{1+\varepsilon}{2}}$, for $t \in [0,1]$.

Theorem 2.10. *(W_n) satisfies LDP with constants $(\log n)$ and rate function I defined by (2.11).*

PROOF. Let $\widetilde{W}_n := \frac{1}{\log n}\int_1^n \delta_{\psi_{[s]}}\frac{ds}{s}$. First we show that $\widetilde{W}_n$ and W_n are exponentially equivalent. By Lemma 2.8 it suffices to verify

$$\lim_{n\to\infty}\frac{1}{\log n}\log \mathbb{P}\{d_\phi(W_n,\widetilde{W}_n) > \varepsilon\} = -\infty$$

Given $f \in \mathcal{C}(\mathcal{C}_\phi, \mathbb{R})$, $\|f\|_L \le \frac{1}{2}$ we have

$$\begin{aligned}
|\int f\,dW_n - \int f\,d\widetilde{W}_n| &= \left|\frac{1}{\log n}\int_1^n f(\psi_{[t]})\,\frac{dt}{t} - \frac{1}{L(n)}\sum_{k=1}^n \frac{1}{k}f(\psi_k)\right| \\
&= \left|\frac{1}{\log n}\sum_{k=1}^{n-1}\log(1+\frac{1}{k})\delta_{\psi_k} - \frac{1}{L(n)}\sum_{k=1}^n \frac{1}{k}f(\psi_k)\right| \\
&\le \left|\frac{1}{L(n)}\log(1+\frac{1}{n}f(\psi_n)\right| + \left|\frac{1}{L(n)}\sum_{k=1}^{n-1}\left(\log(1+\frac{1}{k})-\frac{1}{k}\right)f(\psi_k)\right| \\
&+ \left|\frac{1}{L(n)} - \frac{1}{\log n}\right|\left|\sum_{k=1}^{n-1}\log(1+\frac{1}{k}f(\psi_k)\right| \\
&\le \frac{1}{2}\left(\frac{1}{nL(n)} + \frac{C}{L(n)}\sum_{k=1}^{n-1}\frac{1}{k^2} + \left|1-\frac{\log n}{L(n)}\right|\right)
\end{aligned}$$

which converges to 0 uniformly as $n \to \infty$.

Let $X_1, X_2, ...$ be i.i.d. random variables such that $\mathbb{E}(X_i) = 0$, $\mathbb{E}(X_i^2) = 1$ and the partial sum $S_n = \sum_{i=1}^n X_i$. If one shows that $(\widetilde{W}_n)$ satisfies the LDP with constants $(\log n)$ and rate function I in the case in which X_i are i.i.d. $N(0,1)$-distributed, then the general case follows, see [**3**], i.e., by Skorokhod's representation theorem there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables

(Y_n) and (Z_n) such that $Y_n, n \in \mathbb{N}$ are i.i.d. with the same distribution as X_n, Z_n are i.i.d $N(0,1)$-distributed and $(\widetilde{W}_n)$ corresponding to (Y_n) and respectively (Z_n) are shown to be equivalent with respect to LDP. Therefore, it remains to consider the case for X_i which are i.i.d. $N(0,1)$-distributed.
Let $(\Omega = \mathcal{C}_0[0,\infty), \mathcal{F}, \mathbb{P} = W)$ with coordinate map $X_t(\omega) = \omega(t)$ and $X_i(\omega) := \omega(i) - \omega(i-1) \sim$ i.i.d. $N(0,1)$-distributed. We will show that $(\widetilde{W}_n)$ satisfies the LDP by checking that $\widetilde{W}_n$ and $\widetilde{\mathcal{R}}_n$ are equivalent with respect to LDP.

By Lemma 2.9, this reduces to checking that for any $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\log n} \log W \left\{ \sup_{s\in[n,n+1)} |\theta_s - \psi_n|_\phi > \varepsilon \right\} = -\infty \tag{2.15}$$

The probability $W\{\sup_{s\in[n,n+1)} |\theta_s - \psi_n|_\phi > \varepsilon\}$ is dominated by the sum of three probabilities, $W\{\sup_{s\in[n,n+1)} |\theta_n - \psi_n|_\phi > \frac{\varepsilon}{2}\}$,
$W\{\sup_{s\in[n,n+1)} \sup_{t\in\mathbb{R}_+} \frac{1}{\sqrt{s}} \frac{|\omega(st)-\omega(nt)|}{\phi(t)} > \frac{\varepsilon}{4}\}$,
$W\{\sup_{s\in[n,n+1)} \sup_{t\in\mathbb{R}_+} |\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{n}}| \frac{|\omega(nt)|}{\phi(t)} > \frac{\epsilon}{4}\}$. Given $\varepsilon > 0$, based on arguments of Lemma 4 in [**4**], the estimates below hold for large n and utilize the properties of function ϕ given in Theorem 2.6 along with the properties of Brownian motion,

$$W\left\{\omega \in \mathcal{C}_0[0,\infty) : \sup_{t\in[a,b]} |\omega(t) - \omega(a)| \geq c\right\} \leq 2\exp\left(-\frac{c^2}{2(b-a)}\right)$$

for all $0 \leq a < b$ and $c > 0$, and the inequalities: $\mathrm{Log}(e^{|k|}) \geq \frac{|k|+1}{2}$, $\sup_{s\in[n,n+1)}(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{u}}) \leq \frac{1}{2}n^{-3/2}$. In what follows the positive constant c_{ij} represent the j-th constant in the i-th estimate.

Regarding the first estimate we have

$$\begin{aligned}
&W\left\{\sup_{s\in[n,n+1)} \sup_{t\in\mathbb{R}_+} |\theta_s - \psi_n|_\phi > \frac{\varepsilon}{2}\right\} \leq \\
&W\left\{\sup_{k\geq 1} \sup_{t\in\left[\frac{k}{n},\frac{k+1}{n}\right)} \frac{|\theta_n(t) - \psi_n(t)|}{\phi(t)} > \frac{\varepsilon}{2}\sqrt{n}\right\} \leq \\
&W\left\{\sup_{k\geq 1} \sup_{t\in[k,k+1)} |\omega(t) - \omega(k)| > \frac{\varepsilon}{2}\sqrt{n}\phi\left(\frac{k}{n}\right)\right\}
\end{aligned}$$

Then for sufficiently large n

$$W\left\{\sup_{1\le k\le n}\sup_{t\in[k,k+1)}|\omega(t)-\omega(k)|>\frac{\varepsilon}{2}\sqrt{n}\phi\left(\frac{k}{n}\right)\right\}\le$$
$$\sum_{k=1}^{n}W\left\{\sup_{t\in[k,k+1)}|\omega(t)-\omega(k)|>\frac{\varepsilon}{2}\sqrt{n}\sqrt{\frac{k}{n}}\left(\frac{k}{n}\right)^{\varepsilon}\right\}\le$$
$$\sum_{k=1}^{n}W\left\{\sup_{t\in[k,k+1)}|\omega(t)-\omega(k)|>\frac{\varepsilon}{2}\frac{k^{\frac{1}{2}+\varepsilon}}{n^{\varepsilon}}\right\}\le$$
$$\sum_{k=1}^{n}2\exp\left(-\frac{1}{2}\frac{\varepsilon^2}{4}\frac{k^{1+2\varepsilon}}{n^{2\varepsilon}}\right)\le 2n\exp\left(-\frac{\varepsilon^2}{8}n\right)$$

and

$$W\left\{\sup_{k\ge n+1}\sup_{t\in[k,k+1)}|\omega(t)-\omega(k)|>\frac{\varepsilon}{2}\sqrt{n}\sqrt{\frac{k}{n}\mathrm{Log}\left(\frac{k}{n}\right)}\right\}\le$$
$$\sum_{k=n+1}^{\infty}W\left\{\sup_{t\in[k,k+1)}|\omega(t)-\omega(k)|>\frac{\varepsilon}{2}\sqrt{3k}\right\}\le$$
$$\sum_{k=n+1}^{\infty}2\exp\left(-\frac{1}{2}\frac{\varepsilon^2}{4}9k\right)\le c_{11}e^{-c_{12}n}.$$

Therefore,

$$W\{\sup_{s\in[n,n+1)}|\theta_s-\psi_n|_\phi>\varepsilon\}\le 2n\exp\left(-\frac{\varepsilon^2}{8}n\right)+c_{11}e^{-c_{12}n}.$$

For the second estimate one obtains

$$W\left\{\sup_{s\in[n,n+1)}\sup_{t\in\mathbb{R}_+}\frac{|\omega(st)-\omega(nt)|}{\sqrt{s}\phi(t)}>\frac{\varepsilon}{4}\right\}\le$$
$$\sum_{l\in\mathbb{Z}}W\left\{\sup_{s\in[n,n+1)}\sup_{t\in[e^l,e^{l+1})}\frac{|\omega(st)-\omega(nt)|}{\phi(t)}>\frac{\varepsilon}{4}\sqrt{n}\right\}\le$$
$$\sum_{l\in\mathbb{Z}}W\left\{\sup_{s\in[0,1)}\sup_{t\in[e^l,e^{l+1})}\frac{|\omega((s+n)t)-\omega(nt)|}{\phi(t)}>\frac{\varepsilon}{4}\sqrt{n}\right\}\le$$

$$\sum_{l\in\mathbb{Z}} W\left\{\sup_{s\in[0,1)}\sup_{t\in[1,e)}|\omega((s+n)t)-\omega(nt)|>\frac{\varepsilon}{4}\sqrt{n}\frac{\phi(e^l)}{\sqrt{e^l}}\right\}\leq$$
$$\sum_{l\in\mathbb{Z}} W\left\{\sup_{s\in[0,1)}\sup_{t\in[1,e)}|\omega((s+n)t)-\omega(nt)|>\frac{\varepsilon\sqrt{n\mathrm{Log}(e^{|l|})\zeta(e^{|l|})}}{4}\right\}\leq$$
$$\sum_{l\in\mathbb{Z}} c_{21}n\exp(-c_{22}\varepsilon^2 n(|l|+1))\leq c_{23}n\exp(-c_{24}\varepsilon^2 n),$$

where $\zeta:[1,\infty)\to[1,\infty)$,
$\zeta(t):=\inf_{s\in\mathbb{R}_+\setminus(\frac{1}{t},t)}\frac{\phi(s)^2}{s\mathrm{Log}(s)}$, $\lim_{t\to\infty}\zeta(t)=\infty$.
Finally,

$$W\left\{\sup_{s\in[n,n+1)}\sup_{t\in\mathbb{R}_+}\left|\frac{1}{\sqrt{s}}-\frac{1}{\sqrt{n}}\right|\frac{|\omega(nt)|}{\phi(t)}>\frac{\varepsilon}{4}\right\}\leq$$
$$W\left\{\sup_{t\in\mathbb{R}_+}\frac{|\omega(nt)|}{n\sqrt{n}\phi(t)}>\frac{\varepsilon}{4}\right\}\leq\sum_{l\in\mathbb{Z}} W\left\{\sup_{t\in[e^l,e^{l+1})}\frac{|\omega(t)|}{\phi(t)}>\frac{\varepsilon n}{4}\right\}\leq$$
$$\sum_{l\in\mathbb{Z}} W\left\{\sup_{t\in[1,e)}|w(t)|>\varepsilon\frac{n}{4}\sqrt{\mathrm{Log}(e^{|l|})\zeta(e^{|l|})}\right\}\leq$$
$$\sum_{l\in\mathbb{Z}} c_{31}\exp(-\varepsilon^2 n^2 c_{32}(|l|+1))\leq c_{32}\exp(-c_{34}\varepsilon^2 n^2).$$

which establishes (2.15) and concludes the proof.

□

3. Martingale decomposition of Markov functionals

In this section we show that certain functionals of Markov chains can be written as martingales perturbed by random variables whose maxima (scaled by factor $\frac{1}{\sqrt{n}}$) converge in probability to zero at a prescribed rate. The main interest in such representation stems from the fact that when proving central limit theorem or large deviation principle, the methods in Markovian versus martingale case are essentially different (except when the two cases overlap and either method can be employed), while martingale approach is often preferable. For instance, martingale maximal inequalities offer a useful tool in handling the tail estimates which in turn could be applied to Markovian cases whenever possible.

In what follows we provide criteria for a martingale decomposition and describe the classes of Markov chain functionals that allow such representation.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{B}_E)$ a complete separable metric space, and let $X_n, n \geq 0$ be a Markov chain defined on Ω with values in E. Given a probability measure μ on $(E, \mathcal{B}_E)$, one defines the probability measures $\mathbb{P}_\mu$ by

$$\mathbb{P}_\mu(B) = \mu\mathbb{P}(B) = \int_E \mu(dx)p(x, B), \quad x \in E, B \in \mathcal{F}$$

where $p(x, B)$, $x \in E$, $B \in \mathcal{B}_E$ is the Markov transition function of $X_n, n \geq 0$. We denote by $\mathbb{E}_\mu$ and $\mathbb{E}_x$ the expectations corresponding to $\mathbb{P}_\mu$ and $\mathbb{P}_x$ respectively. Inductively one defines the n-step transition probability by $p^n(x, B) = \mathbb{P}(X_n \in B | X_0 = x) = \mathbb{P}(X_{n+m} \in B | X_m = x)$.

Let P be the transition probability operator defined as

$$P\varphi(x) := \mathbb{E}[\varphi(x_{n+1})|x_n = x] = \int_E p(x, dy)\varphi(y)$$

and denote by P^n the n-step transition operator corresponding to the n-step transition probability $p^n(x, B)$.

Theorem 3.1. *Let $X_n, n \geq 0$ be an ergodic E-valued Markov chain with initial distribution μ and unique invariant probability measure m.*
If $g \in L^2(m) := \{g : E \to \mathbb{R} : \int_E g^2\, dm < \infty\}$ satisfies the properties:

(i) $\int_E g\, dm = 0$
(ii) $\|P^k g\|_{L^2(m)} \leq \rho^k \|g\|_{L^2(m)}$ *for some* $0 < \rho < 1$, $k \in \mathbb{N}$
(iii) $\frac{d\mu P^k}{dm} \leq D < \infty$, $k \in \mathbb{N}$ *and* $\int_{\{x:\, g^2(x) > n\}} g^2(x)m(dx) \leq \exp(-\varphi(n))$ *for n large, with* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{x\to\infty} \frac{\varphi(x)}{\log(x)} \to \infty$
(iv) $|P^k g(x)| \leq Cn$, *whenever* $|g(x)| \leq n$, *for some* $1 < C < \infty$, $x \in E$, $k \in \mathbb{N}$, *and large n*

then

$$S_n(g) := \sum_{k=0}^{n-1} g(X_k) = M_n + R_n \tag{3.1}$$

where M_n is a mean zero martingale relative to $(\Omega, \mathcal{F}_n, \mathbb{P})$, with the natural filtration $\mathcal{F}_n = \sigma(X_0, ..., X_n)$ and

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left\{ \frac{\sup_{1\le k\le n} R_k^2}{n} > \varepsilon \right\} = -\infty. \tag{3.2}$$

PROOF. We show that under our hypothesis the Poisson equation $(I-P)u = g$ has a unique solution. Notice that $I-P$ is not invertible, while $((1+\varepsilon)I - P)u_\varepsilon = g$, $\varepsilon > 0$ has unique solution

$$u_\varepsilon = ((1+\varepsilon)I - P)^{-1} g = \frac{1}{1+\varepsilon} \sum_{k=1}^{\infty} \frac{P^{k-1} g}{(1+\varepsilon)^{k-1}} \tag{3.3}$$

thanks to $1+\varepsilon$ being in the resolvent of $L^2(m)$-contractive P, whose spectral radius is 1. Condition (ii) implies that the series in (3.3) converges in $L^2(m)$ and we have

$$u(x) := \lim_{\varepsilon\to 0} u_\varepsilon(x) = \sum_{i=0}^{\infty} P^k g. \tag{3.4}$$

Therefore, $S_n(g) = \sum_{k=0}^{n-1} g(X_k) = \sum_{k=1}^{n} (u(X_k) - Pu(X_{k-1})) + u(X_0) - u(X_n)$ and by Markov property

$$M_n := \sum_{k=1}^{n} (u(X_k) - Pu(X_{k-1}))$$

is a mean zero martingale in $L^2(m)$ with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, ..., X_n)$. Setting

$$R_n := u(X_0) - u(X_n) \tag{3.5}$$

establishes the martingale decomposition (3.1).

To simplify notation, summation index below and whenever else integer is needed, is understood as the integer part of the number. For each $k \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{P}(u^2(X_k) > Cn) \ &\le\ \mathbb{P}\left(\left| \sum_{i=0}^{\sqrt[4]{n}} P^i g(X_k) \right| > \frac{\sqrt{Cn}}{2} \right) \\ &+\ \mathbb{P}\left(\left| \sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k) \right| > \frac{\sqrt{Cn}}{2} \right) \end{aligned}$$

The second term with $n \geq 4$ satisfies

$$\mathbb{P}\left(\left|\sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k)\right| > \frac{\sqrt{Cn}}{2}\right) \leq \mathbb{E}\left|\sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k)\right| =$$

$$\left\|\sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k)\right\|_{L^1(\Omega)} \leq \left\|\sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k)\right\|_{L^2(\Omega)} \leq$$

$$\sum_{i=\sqrt[4]{n}+1}^{\infty} \|P^i g(X_k)\|_{L^2(\Omega)}$$

Using conditions (iii) and (ii) we get

$$\|P^i g(X_k)\|^2_{L^2(\Omega)} = \mathbb{E}(P^i g(X_k))^2 = \mathbb{E}\left(\int g(y) p^i(X_k, dy)\right)^2 =$$

$$\int\left(\int g(y)p^i(x,dy)\right)^2 \mu P^k(dx) \leq D\|P^i g(x)\|^2_{L^2(m)} \leq D\rho^{2i}\|g(x)\|^2_{L^2(m)}$$

Consequently, for n sufficiently large,

$$\mathbb{P}\left(\left|\sum_{i=\sqrt[4]{n}+1}^{\infty} P^i g(X_k)\right| > \frac{\sqrt{Cn}}{2}\right) \leq \sqrt{D}\frac{\rho^{\sqrt[4]{n}+1}}{1-\rho}\|g\|_{L^2(m)} \leq A\exp(-B\sqrt[4]{n})$$

for some positive constants A and B. Turning to the first term, we have

$$\mathbb{P}\left(\left|\sum_{i=0}^{\sqrt[4]{n}} P^i g(X_k)\right| > \frac{\sqrt{Cn}}{2}\right) \leq \sum_{i=0}^{\sqrt[4]{n}} \mathbb{P}\left(|P^i g(X_k)|^2 > \frac{Cn}{4}\right),$$

and for $\Omega = \{g^2(X_k) > \frac{n}{4}\} \cup \{g^2(X_k) \leq \frac{n}{4}\}$ gives

$$\begin{aligned}\mathbb{P}\left(|P^i g(X_k)|^2 > \frac{Cn}{4}\right) &\leq \mathbb{P}(g^2(X_k) > \frac{n}{4}) \\ &+ \mathbb{P}\left(\{g^2(X_k) \leq \frac{n}{4}\} \cap \{|P^i g(X_k)|^2 > \frac{Cn}{4}\}\right)\end{aligned}$$

and (iv) makes the second term disappear while the first term, thanks to (ii), satisfies

$$\mathbb{P}\left(g^2(X_k) > \frac{Cn}{4}\right) = \mu\mathbb{P}^k\left(g^2 > \frac{Cn}{4}\right) \leq \exp\left(-\varphi\left(\frac{n}{4}\right)\right).$$

Combining the above yields

$$
\begin{aligned}
\mathbb{P}(\max_{1\le k\le n}|R_k| > \sqrt{\varepsilon C n}) &\le n\mathbb{P}(|R_k| > \sqrt{\varepsilon C n}) \le 2n\mathbb{P}(|u(X_k)| > \frac{\sqrt{\varepsilon C n}}{2}) \\
&\le 2n\left[A\exp\left(-B\sqrt[4]{n}\right) + \sqrt[4]{n}\exp(-\varphi\left(\frac{n}{4}\right))\right]
\end{aligned}
$$

and (3.2) follows. □

Theorem 3.2. *The following classes of processes satisfy the conclusions* (3.1) *and* (3.2) *of theorem* (3.1)*:*

(a) *finite state irreducible, aperiodic Markov chains*
(b) *uniformly ergodic Markov chains with bounded functions* g
(c) *Markov chains obtained by quantization* P^n *of continuous time Markov processes* P^t*, that are symmetric on* $L^2(m)$ *(i.e. for which* m *is a reversible measure).*

Proof. (a) Since there is a unique invariant measure $m = (m_1, ..., m_N)$, the assumption (i) for $g = (g_1, ..., g_N)$ can be written as $\sum_{i=1}^N g_i m_i = 0$. Then by the Fredholm Alternative, there is a solution of the Poisson equation $(I - P)u \equiv Au = g$ if and only if g is orthogonal to w where $A^* w \equiv A^T w = 0$. The later is true because $mA = 0$ or $A^T m^T = 0$. We do not need to verify (ii) since that was only used to prove the existence of solution to the Poisson equation. Also, since g on $\{1, ..., N\}$ is bounded, (iii) and(iv) clearly hold.

(b) If the Markov chain is uniformly ergodic then the operator $Q := I - P$ is normally solvable and condition (i) enables the uniqueness of the solution of the Poisson equation. That is $u = R_0 g$ where R_0 is the potential operator defined by $R_0 := \sum_{n=0}^{\infty}[P^n - \Pi]$ and the projection operator in $B(E)$ is defined by $\Pi g(x) := \int_E m(dx)g(y)\mathbb{1}(x)$. Since R_0 is a bounded operator conclusions (3.1) and (3.2) follows directly. Moreover, the conditions (iii) and (iv) holds because g is bounded.

(c) For the sake of the discussion, we consider a class of continuous time Markov processes in $(\mathbb{R}^d, |\cdot|)$ such that $P^t g = e^{tA} g$ for smooth functions g, where the infinitesimal generator A has a discrete spectrum $\{-\lambda_n, n \in \mathbb{N}\}$ with $\lambda_0 = 0 < \lambda_1 < \lambda_2 \cdots$ and its corresponding orthonormal in $L^2(m)$

set of eigenfunctions $\{e_n, n \in \mathbb{N}\}$ with $e_0 = 1$. Then for any g in the orthogonal complement $1^\perp$, the condition (i) is satisfied and $P^k e_n = e^{-\lambda_n k} e_n$, $n \geq 1$. Consequently, for $g \in 1^\perp$, $g = \sum_{n=1}^\infty \alpha_n e_n$, $P^k g(x) = \sum_{n=1}^\infty \alpha_n e^{-\lambda_n k} e_n$, whence $\|P^k g(x)\|_{L^2(m)} \leq \rho^k \|g\|_{L^2(m)}$ with $0 < \rho = e^{-\lambda_1} < 1$ and this gives (ii). Furthermore, the invariant measure m often has a density with respect to Lebesque measure and condition $\frac{d\mu P^k}{dm} \leq D < \infty$ is satisfied. Finally, given m, the tail condition $\int_{\{g^2>n\}} g^2\, dm \leq \exp(-\varphi(n))$ is readily verifiable for a large class of $g \in 1^\perp$ in addition to which g is chosen to satisfy (iv). We remark that for any finite linear combination of eigenfunctions from $1^\perp$, the condition (iv) is redundant and the tail condition reduces to $\int_{\{e_i^2>a\}} e_i^2\, dm \leq e^{-\varphi(a)}$ for large a. Namely, given $g = \sum_{i=1}^N \alpha_i e_i$,

$$|P^i g(x)|^2 \leq \|g\|^2_{L^2(m)}(e_1^2(x) + \cdots + e_N^2(x))$$

and therefore

$$\mathbb{P}(|P^i g(X_k)|^2 > n) \leq \mathbb{P}\left(\sum_{i=1}^N e_i^2(X_k) > n\right) \leq$$

$$\sum_{i=1}^N \mathbb{P}\left(e_i^2(X_k) > \frac{n}{N}\right) \leq ND \max_{1\leq i\leq N} \int_{\{e_i^2>n/N\}} e_i^2 dm$$

as claimed.

□

Hypercontractivity Example

Consider a 1-dimensional Ornstein-Uhlenbeck process X_t satisfying the equation $dX_t = -\frac{1}{2}X_t dt + dB_t$, $X_0 = x$, where B_t is the standard Brownian motion, with infinitesimal generator $A = \frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x\frac{d}{dx}$ and the invariant measure $dm = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Then $P^t g(x) := \int g(y) p^t(x,y) dy$, where

$$p^t(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-t})}} e^{-\frac{(xe^{-\frac{t}{2}}-y)^2}{2(1-e^{-t})}},$$

is a m-symmetric hypercontractive Hermite semigroup with other examples, including Laguerre semigroup, studied in [**7**] and [**6**].

Here, for Hermite polynomials $H_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{n}}$, we have $H_0 = 1, H_1 = x, H_2 = x^2 - 1, H_3 = x^3 - 3x, \ldots$, $e_n = \frac{H_n}{\sqrt{n!}}$ form an orthonormal basis for $L^2(m)$ and $AH_n = -\frac{n}{2} H_n$ gives the corresponding eigenvalues $\lambda_n = -\frac{n}{2}$, $n \in \mathbb{N}$. Letting $t = 0, 1, 2, \ldots$ we obtain the embedded Markov chain X_n. Since

$$d\mu P^k = \frac{1}{\sqrt{2\pi(1-e^{-k})}} e^{-\frac{(xe^{-\frac{k}{2}}-y)^2}{2(1-e^{-k})}} \, dy$$

then, for example, taking $\mu = \delta_x = \delta_0$ we have

$$\frac{d\mu P^k}{dm} = \frac{1}{\sqrt{1-e^{-k}}} e^{-\frac{y^2}{2}\left(\frac{e^{-k}}{1-e^{-k}}\right)} \leq \sqrt{\frac{e}{e-1}} \equiv D$$

It is easy to see that for polynomial domination, $|g(x)| \leq B|x|^l$, $l \in \mathbb{N}$, and for large n

$$\int_{\{g^2 > n\}} g^2 \, dm \leq C n^{2l-1} e^{-\frac{n^2}{2}} \leq e^{-\frac{n^2}{4}}$$

whence one may choose $\varphi(x) = \frac{x^2}{4}$ to satisfy the tail estimate.
For example, taking $g(x) = x$ one gets

$$\begin{aligned} |P^i g(x)|^2 &= \left(\int |g(y) p^i(x,y) \, dy| \right)^2 \leq D \|g\|_{L^2(m)} = D^2(1 - e^{-i} + x^2) \\ &\leq 4D \max(1, x^2) \end{aligned}$$

and condition (iv) holds true with $C = 2D$.
For general g, even growing exponentially fast to infinity, it requires some work through estimates and usually g determines φ.

4. Large deviation principle for additive functionals

Let X_n, $n \geq 0$ be an ergodic E-valued Markov chain with stationary measure m, as in Theorem 3.1. Let $S_n = \sum_{k=0}^{n-1} g(X_k)$ and $\mathbb{E}_m(S_1^2) = \sigma^2 \in (0, \infty)$. According to Theorem 3.1, $S_n = M_n + R_n$, where M_n is a mean zero martingale and R_n satisfies (3.2). Let W_n and W_n^M be the empirical measures defined in (2.4) and (2.5) corresponding to the interpolation processes Ψ_n and Ψ_n^M defined by (2.3) and (2.6) respectively. By Theorem 2.2, W_n converges weakly to the Wiener measure W on $\mathcal{C}[0, \infty)$. To conclude our analysis we invoke a martingale LDP [**5**], applied here to W_n^M, and show that W_n^M and W_n are LDP equivalent.

Lemma 4.1. *If*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{|\Psi_n - \Psi_n^M|_\phi > \varepsilon\} = -\infty \tag{4.1}$$

then $(W_n) \equiv (\frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} \delta_{\Psi_k \in \cdot})$ *and* $(W_n^M) \equiv (\frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} \delta_{\Psi_k^M \in \cdot})$ *are equivalent with respect to LDP.*

Proof. By the above Lemma, we need to verify

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{d_\phi(W_n, W_n^M) > \varepsilon\} = -\infty \tag{4.2}$$

where d_ϕ is defined in (2.13). For $f \in \mathcal{C}(\mathcal{C}_\phi, \mathbb{R})$, $\|f\|_L \le \frac{1}{2}$ we have

$$\left| \int f\, dW_n - \int f\, dW_n^M \right| \le \frac{1}{L(n)} \sum_{k=1}^n \frac{1}{k} |f(\Psi_k) - f(\Psi_k^M)| =$$

$$\frac{1}{L(n)} \sum_{k=1}^{[n^{\varepsilon/2}]} \frac{1}{k} |f(\Psi_k) - f(\Psi_k^M)| + \frac{1}{L(n)} \sum_{k=[n^{\varepsilon/2}]+1}^{n} \frac{1}{k} \frac{|f(\Psi_k) - f(\Psi_k^M)|}{|\Psi_k - \Psi_k^M|_\phi}$$

$$|\Psi_k - \Psi_k^M|_\phi \le \frac{L([n^{\varepsilon/2}]}{L(n)} + \frac{L(n) - L([n^{\varepsilon/2}])}{2L(n)} \sup_{1+[n^{\varepsilon/2}]\le k\le n} |\Psi_k - \Psi_k^M|_\phi$$

$$\le \frac{\varepsilon}{2} + \frac{1}{2} \sup_{1+[n^{\varepsilon/2}]\le k\le n} |\Psi_k - \Psi_k^M|_\phi$$

whence

$$\mathbb{P}\{d_\phi(W_n, W_n^M) > \varepsilon\} \le \mathbb{P}\{\sup_{1+[n^{\varepsilon/2}]\le k\le n} |\Psi_k - \Psi_k^M|_\phi > \varepsilon\}.$$

Condition (4.1) implies that for every $\varepsilon > 0$ and $N > 0$, there exists k_0 such that for any $k \ge k_0$, $|\Psi_k - \Psi_k^M|_\phi < k^{-\frac{2N}{\varepsilon}} k^{-2}$. Therefore,

$$\mathbb{P}\{d_\phi(W_n, W_n^M) > \varepsilon\} \le \sum_{k=[n^{\varepsilon/2}]+1}^{n} \mathbb{P}\{|\Psi_k - \Psi_k^M|_\phi > \varepsilon\} \le$$

$$\sum_{k=[n^{\varepsilon/2}]+1}^{n} k^{-\frac{2N}{\varepsilon}} k^{-2} \le n^{-N} \sum_{k=1}^{\infty} k^{-2} = cn^{-N}.$$

where c is a positive constant and (4.2) holds. □

Assume in addition to the conditions of Theorem 3.1 that $\sum_{k=n}^{\infty} \exp(-\varphi(k)) < \frac{1}{n^\gamma}$, for any positive γ, for large n (for example, $\varphi(x) \sim x^\alpha$, $\alpha > 0$).

Theorem 4.2. *The sequence* (W_n) *satisfies the large deviation principle with constants* $(\log n)$ *and rate function* $I|_{\mathcal{M}_1(\mathcal{C}_\phi)}$ *defined in* (2.11)*, that is, for any Borel set* $A \subseteq \mathcal{M}_1(\mathcal{C}_\phi)$*,*

$$\begin{aligned} -\inf_{A^\circ} I &\leq \liminf_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{W_n \in A\} \\ &\leq \limsup_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\{W_n \in A\} \leq -\inf_{\bar{A}} I \end{aligned}$$

where $\mathcal{M}_1(\mathcal{C}_\phi) := \{Q \in \mathcal{M}_1(\mathcal{C}[0,\infty)) : Q(\mathcal{C}_\phi) = 1\}$*, and* ϕ *is defined in* (2.9).

Proof. By Lemma 4.1 it suffices to check that (4.1) holds. We have

$$\mathbb{P}\{|\Psi_k - \Psi_k^M|_\phi > \varepsilon\} = \mathbb{P}\left\{\sup_{t\in\mathbf{R}_+} \frac{|\Psi_k(t) - \Psi_k^M(t)|}{\phi(t)} > \varepsilon\right\} =$$

$$\mathbb{P}\left\{\sup_{1\leq k<\infty} \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} \frac{1}{\sigma\sqrt{n}} \frac{|R_{[nt]} - (nt-[nt])(R_{[nt]+1} - R_{[nt]}|}{\phi(t)} > \varepsilon\right\}$$

$$\leq \mathbb{P}\left\{\sup_{1\leq k<\infty} |R_k| > \varepsilon\sigma\sqrt{n}\phi(\frac{k}{n})\right\} \leq \sum_{k=1}^{n} \mathbb{P}\left\{|R_k| > \varepsilon\sigma\sqrt{n}\sqrt{\left(\frac{k}{n}\right)^\varepsilon}\right\}$$

$$+ \sum_{k=n+1}^{\infty} \mathbb{P}\left\{|R_k| > \varepsilon\sigma\sqrt{n}\sqrt{\frac{k}{n} Log(\frac{k}{n})}\right\} \leq n\mathbb{P}\{|R_k| > \varepsilon\sigma\sqrt{n}\} +$$

$$\sum_{k=n+1}^{\infty} \mathbb{P}\{|R_k| > \varepsilon\sigma\sqrt{3k}\} \leq c_{41} n \exp(-c_{42}\sqrt[4]{n}) + c_{43} n \exp(-\varphi(n)) + n^{-\gamma}$$

which ends the proof.

□

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