International Journal of Computational and Theoretical Statistics
ISSN (2210-1519)
Int. J. Comp. Theo. Stat. 4, No. 1 (May-2017)

An Almost Sure Central Limit Theorem for Autoregressive Processes

Adina Oprisan[1]

[1] Department of Mathematics and Statistics, Canisius College, Buffalo NY, USA

Received February 10, 2017, Revised April 1, 2017, Accepted April 10, 2017, Published May 1, 2017

Abstract: We consider additive functionals for a class of autoregressive processes and prove that their empirical measures with logarithmic averaging converge almost surely via a martingale decomposition. A method developed in our previous work on additive functionals of ergodic Markov processes is applied in the particular case of autoregressive processes that are uniformly ergodic. We also obtain some equivalent results on almost sure asymptotic behavior.

Keywords: Almost sure central limit theorem, Autoregressive process, Empirical process, Ergodicity, Martingale decomposition

1. Introduction

Consider the first-order autoregressive process defined as: $X_n = \theta X_{n-1} + \varepsilon_n$, for n ≥ 1, with initial state $X_0 = \varepsilon_0$ and the driven noise $\{\varepsilon_n\}_{n\geq 0}$ defined as a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a complete separable metric space $(E, \mathcal{B}_E)$. Assume that the random variable ε_0 has the probability distribution μ and the random variables $\{\varepsilon_n\}_{n\geq 1}$ are independent and identically distributed, independent of ε_0, with mean zero and finite variance σ^2 and with probability distribution ν.

Thus $\{X_n, n \geq 0\}$ is a Markov chain defined on Ω with values in the general Polish space $(E, \mathcal{B}_E)$ and with initial probability measure μ. Let $p(x, A)$ be its transition probability kernel,

$$p(x, A) = P(X_{n+1} \in A \mid X_n = x) = P(\theta x + \varepsilon_n \in A) = \nu(A - \theta x).$$

Then one defines the transition probability operator on the space of bounded continuous functions, $P: \mathcal{C}_b(E) \to \mathcal{C}_b(E)$, by

$$P\varphi(x) := \mathbb{E}[\varphi(X_{n+1}) | X_n = x] = \int p(x, dy)\varphi(y).$$

Therefore,

$$\mathrm{P}\varphi\,(\mathrm{x}) = \int \varphi\,(\mathrm{y} + \theta \mathrm{x})\nu(\mathrm{dy}).$$

Let $p^n(x, A) := P(X_n \in A \,| X_0 = x)$ be the n-step transition probability and P^n its corresponding transition operator. For initial measure μ one can define the probability measure $\mu P(A) = P_\mu(A) = \int \mu(dx) p(x, A)$.

Throughout this paper, we assume that the autoregressive process $\{X_n, n \geq 0\}$ satisfies the following properties:

A1: the process is stable, i.e. $|\theta| < 1$;

A2: the process is smooth, i.e. the distribution of ε_n has a continuous density g,

E-mail address: oprisana@canisius.edu

$p(x, dy) = g(y - \theta x)dy,\ \ (x, y) \to g(y - \theta x)$ continuous;

A3: the process is simple, i.e. the density is positive $\forall\, x \in \mathbb{R}$;

A4: $E\, log^+|\varepsilon_n| < \infty,\ log^+(x) = \max\{0, \log x\}$;

A5: Doeblin condition: for some $m > 0, \varepsilon < 1, \delta > 0$, there exists a probability measure ϕ such that if $\phi(A) > 0$ then for any $x \in E, P^m(x, A) \geq \delta$.

Proposition 1.1 The autoregressive process $\{X_n, n \geq 0\}$ satisfying the properties A1 − A5 is uniformly ergodic.

Proof: First, note that the Markov chain $\{X_n, n \geq 0\}$ is a Harris chain (see [6], chapter 5, example 6.2). From A4, we get that $\mathbb{E}|\varepsilon_n| < \infty$, and therefore one can chose M sufficiently large such that $\mathbb{E}|\varepsilon_1| \leq (1-\theta)M$. Thus, if $|x| \geq M$,

$$\mathbb{E}_x|X_1| \leq |\theta||x| + (1 - |\theta|)M \leq |x|.$$

Therefore, according to Theorem (3.9) [6], the Markov chain is Harris recurrent, and so it admits a stationary measure. Next, we want to show that the stationary measure is unique.

Let's define $S_n = \sum_{k=1}^{n} \theta^k \varepsilon_k$ and the filtration $\mathcal{F}_n = \sigma\{\varepsilon_k, k \leq n\}$. Then $\{S_n, n \geq 0\}$ is a $\mathcal{F}_n$-martingale. Since $\mathbb{E}|S_n|^2 \leq \sigma^2 \sum_{k=1}^{n} |\theta|^{2k} < \infty$ and $sup_n \mathbb{E}|S_n|^2 = \sigma^2 \sum_{k=1}^{n} |\theta|^{2k} < \infty$, the sum S_n is an uniformly integrable martingale, so it converges a.s. to a random variable X that defines a stationary measure π.

On the other hand, $X_n = (\varepsilon_n + \theta\varepsilon_{n-1} + \cdots + \theta^{n-1}\varepsilon_1) + \theta^n X_0$. While the first part converges a.s. to X and the other converges in probability to 0, it follows that the law of X_n converges weakly to the stationary distribution π. Thus, the stationary measure is unique.

Since the Markov chain is aperiodic and Harris recurrent then the Doeblin condition (A5) is equivalent with the uniform ergodicity of the Markov chain (Theorem 16.2.3 in [12]).

Note that the Markov chain $\{X_n, n \geq 0\}$ is uniformly ergodic if

$$sup_{\|\varphi\| \leq 1} \| (P^n - \Pi)\varphi \| \to 0,\ \ n \to \infty,$$

where the stationary projector $\Pi: \mathcal{B}_E \to \mathcal{B}_E$ is defined as

$$\Pi\varphi(x) = \int \pi(dy)\varphi(y)1(x),\quad 1(x) = 1 \text{ for all } x \in E.$$

Π is a projector operator since $\Pi^2 = \Pi$.

2. Martingale Decomposition for Additive Functionals of Autoregressive Processes

Let's assume that $\{X_n, n \geq 0\}$ is an autoregressive process with unique invariant measure π so that

$$\pi(A) = \int p(x, A)\pi(dx).$$

Define the additive functional

$$S_n(f) = \sum_{k=0}^{n-1} f(X_k)\,, n \geq 1, \tag{2.1}$$

where $f \in L_0^2(\pi) = \{f: E \to \mathbb{R}, \| f \|_2 = (\int |f|^2 d\pi)^{\frac{1}{2}} < \infty, \int f\, d\pi = 0\}$ such that $\mathbb{E}_\pi(S_1^2) = \sigma^2 \in (0, \infty)$ and for any initial distribution μ and any $k \geq 1$, $\mathbb{E}_\mu(S_k^2) < \infty$.

Theorem 2.1 *Let $\{X_n, n \geq 0\}$ be an autoregressive process satisfying the conditions (A1)-(A5) and π it unique invariant measure. Let $f \in L_0^2(\pi)$ such that:*

(i) $\| P^k f \|_{L^2(\pi)} \leq \rho^k\ \| f \|_{L^2(\pi)}$ for some $0 < \rho < 1, k \in \mathbb{N}$;

(ii) $\frac{d\mu P^k}{d\pi} \leq D < \infty,\ \ k \in \mathbb{N}$ and $\int 1_{\{x: f^2(x) > n\}} f^2(x)\pi(dx) \leq \exp(-\varphi(n))$ for n large, with

$\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{x\to\infty} \frac{\varphi(x)}{\log(x)} \to \infty$;

(iii) $|P^k f(x)| \le Cn$, *whenever* $|f(x)| \le n$, *for some* $1 < C < \infty, x \in E, k \in \mathbb{N}$, *and large n.*

Then the additive functional $S_n(f)$ *defined in (2.1) satisfies the martingale decomposition:*

$$S_n(f) = M_n + R_n ,$$

where $\{M_n, n \ge 1\}$ *is a mean zero martingale relative to* $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, *the natural filtration generated by the trajectories of the Markov chain. The remainder* R_n *converges in probability to zero and*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P} \left\{ \frac{sup_{1\le k\le n} R_k^2}{n} > \varepsilon \right\} = -\infty. \qquad (2.2)$$

Proof: The Markov property of the autoregressive process can be represented in the form

$$\mathbb{E}[\varphi(X_n)|\mathcal{F}_{n-1}] = P\varphi(X_{n-1}).$$

Let M_n be the sum of martingale differences,

$$M_n = \sum_{k=1}^{n} (\varphi(X_k) - \mathbb{E}[\varphi(X_k)|\mathcal{F}_{k-1}]),$$

then

$$M_n = \varphi(X_n) - \varphi(X_0) + \sum_{k=1}^{n-1} [I - P]\varphi(X_k). \qquad (2.3)$$

As proved in Proposition (1.1), under the conditions (A1)-(A5) the autoregressive process is uniformly ergodic, therefore its infinitesimal generator $I-P$ is reducible-invertible operator (Theorem 2.2.2 in [7]). In this case, the Poisson equation

$$[I - P]\varphi(x) = f$$

admits a unique solution. The solution is the potential operator of $I - P$, denoted by R_0, which is a bounded operator with the following representation:

$$R_0 = \sum_{n=0}^{\infty} [P^n - \Pi], \qquad where \quad \Pi f(x) = \int \pi(dy) f(y) 1(x).$$

The assumption $\Pi f = 0$ implies that $R_0 f = \sum_{n=0}^{\infty} P^n f$ and the equation (2.3) yields the martingale decomposition:

$$S_n(f) = M_n + R_n ,$$

with $\{M_n, n \ge 1\}$ a mean-zero martingale with respect to the filtration $\mathcal{F}_n$ and the remainder term

$$R_n = \varphi(X_0) - \varphi(X_n) = R_0 f(X_0) - R_0 f(X_n).$$

In [8] we proved a martingale decomposition for additive functionals of ergodic Markov processes. Note that the Poisson equation in the martingale decomposition theorem in [8] does not have unique solution (the operator $I - P$ is not invertible) but it was proved that the limit of the perturbed operator have the same representation, $\sum_{n=0}^{\infty} P^n f$.
By following the lines of Theorem 3.1. in [8], it can be shown that the remainder R_n converges in probability to zero at the rate given by (2.2).

Corollary 2.2 If $\{X_n, n \ge 0\}$ is a uniformly ergodic autoregressive process with unique invariant measure π and if f is a bounded function such that $\int f(x)\pi(dx) = 0$ then the additive functional $S_n(f)$ satisfies the martingale decomposition in Theorem 2.1.

3. Functional almost sure central limit theorem

Almost sure central limit theorems have been extensively studied, starting with results for sequences of independent and identically distributed random variables, established by Brosamler [3] (1988), Shatte [14] (1988,1991) and Lacey and Phillip [9] (1990). A universal almost sure central limit theorem was considered by Berkes and Csáki in [2], and provides limit theorems for partial sums, extremes, empirical distributions and local times. Important results for martingales are due to Maaouia [11], Chaabane [4], Chaabane and Maaouia [5], Lifshits [10] and Bercu [1].

We consider additive functionals of autoregressive processes $S_n(f)$ defined in (2.1). The functional central limit theorem, almost sure central limit theorems and large deviations for additive functionals of Markov processes in both discrete and continuous time setting, we have previously studied in [8] and [13]. Due to the martingale decomposition, Theorem 2.1, we can apply the method developed in [8] and obtain the almost sure central limit theorem for autoregressive processes.

Let's define the interpolation process

$$\Psi_t^n(\omega) := \frac{1}{\sigma\sqrt{n}}\left(S_{[nt]} + (nt - [nt])\left(S_{[nt]+1} - S_{[nt]}\right)\right), \qquad 0 \le t < \infty, \tag{3.1}$$

and the empirical process $W_n: \mathcal{C}[0,\infty) \to \mathcal{M}\left(\mathcal{C}[0,\infty)\right)$,

$$W_n(\omega, A) := \frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}\delta_{\{\Psi^k(\omega)\}}(A). \tag{3.2}$$

Theorem 3.1 (FASCLT) The sequence of empirical processes W_n converges weakly to the Wiener measure $\mathbb{W}$ on $\mathcal{C}[0,\infty)$, for almost every $\omega \in \Omega$.

FASCLT implies that for any bounded measurable function h, continuous a.e.,

$$\lim_{n\to\infty}\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}h(\Psi^k(\cdot,\omega)) = \mathbb{E}h(\mathbb{W}(\cdot)) = \int h(x)d\mathbb{W}(x), \qquad P-a.s.$$

Corollary 3.2 Let $\Phi_t^n(\omega) := \frac{1}{\sigma\sqrt{n}}S_{nt}$. Then:

(i)

$$\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}h(\Phi^k(\cdot,\omega)) \to \int h(x)d\mathbb{W}(x), \quad P-a.s.$$

(ii)

$$\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}h\left(\frac{S_k}{\sigma\sqrt{k}}\right) \to \int h(x)d\mathbb{W}(x), \quad P-a.s.$$

Proof: Since $\mathbb{E}(S_1^2) < \infty$, we get $\| \Psi^n - \Phi^n \| \le \frac{1}{\sqrt{n}} max_{1\le i\le n}|f(X_i)| \to 0$ a.s. , so (i) is proved. (ii) is the particular case when $t = 1$, i.e. the functional almost sure CLT for a sequence of random variables that are functions of autoregressive processes. This is in agreement with the celebrated almost sure CLT established in [3], [14], [9].

Corollary 3.3 The following asymptotic results follows:

(i)

$$\lim_{n\to\infty}\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}1_{(-\infty,\alpha]}\left(\frac{1}{\sigma\sqrt{k}}S_{kt}\right) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\alpha} e^{-u^2/2t}du \qquad \mathbb{P}-a.s.$$

(ii)

$$\lim_{n\to\infty}\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}1_{[\alpha,\infty)}\left(sup_{0\le t\le T}\left(\frac{1}{\sigma\sqrt{k}}S_{kt}\right)\right) = \frac{2}{\sqrt{2\pi T}}\int_{\alpha}^{\infty} e^{-u^2/2T}du \qquad \mathbb{P}-a.s.$$

Proof: (i) follows directly from Corollary 3.2.(ii), by taking $h(x) = 1_A(x)$, with $A = \{x \in \mathcal{C}[0,\infty): x(t) \le \alpha\}$ for any $\alpha \in \mathbb{R}$. Similarly, (ii) follows by taking $A = \{x \in \mathcal{C}[0,\infty): sup_{0\le t\le T}x(t) \ge \alpha\}$ and applying the reflection principle of Brownian motion.

Remark 3.4 For an autoregressive process $X_n = \theta^n \varepsilon_0 + \sum_{j=1}^{n} \theta^{n-j} \varepsilon_j$, it is known that the least-squares estimator for the unknown parameter θ is

$$\hat{\theta}_n = \frac{\sum_{k=1}^{n} X_{k-1} X_k}{\sum_{k=1}^{n} X_{k-1}^2} ,$$

and

$$\hat{\theta}_n - \theta = \frac{\sum_{k=1}^{n} X_{k-1} \varepsilon_k}{\sum_{k=1}^{n} X_{k-1}^2} .$$

If $|\theta| < 1$ then the estimator $\hat{\theta}_n$ converges almost surely to θ and

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \Longrightarrow \mathcal{N}(0,1-\theta^2). \qquad (3.3)$$

Let $M_n = \sum_{k=1}^{n} X_{k-1} \varepsilon_k$, $s_n = \sum_{k=0}^{n} X_k^2$ and $f_n = \frac{X_n^2}{s_n}$. Since $\{M_n, n \geq 1\}$ is a martingale, the almost sure central limit theorem for martingales proved in [4] yields

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k h\left(\frac{M_k}{\sqrt{s_{k-1}}}\right) = \int h(x) dG(x) \qquad a.s., \qquad (3.4)$$

which is the almost sure version of the central limit theorem (3.3).

In conclusion, our result in Corollary 3.2 generalizes (3.4) and provides an almost sure version for the invariance principle of autoregressive processes.

References

[1] B. Bercu, “On the convergence of moments in the almost sure central limit theorem for martingales with statistical applications,” Stochastic Processes and their Applications, vol. 111, pp. 157–173, 2004.

[2] I. Berkes and E. Csáki, “A universal result in almost sure central limit theorem”, Stochastic Process. Appl., vol. 94, pp. 105–134, 2001.

[3] G.A. Brosamler, “An almost everywhere central limit theorem”, Math. Proc. Cambridge Phil. Soc., vol. 104, pp. 561–574, 1998.

[4] F. Chaabane, “Version forte du théorème de la limite centrale fonctionnel pour les martingales”, C.R. Acad. Sci. Paris, vol. 323 pp. 195–198, 1996.

[5] F. Chaabane and F Maaouia, “Theoremes limites avec poids pour les martingales vectorielles”, Essaim Prob. Stat., vol. 4, pp. 137–189, 2000.

[6] Richard Durrett, “Probability: Theory and Examples”, 2nd ed., Duxbury Press, 1996.

[7] V.S. Korolyuk and A.F. Turbin, “Mathematical Foundations of the State Lumping of Large Systems”, Kluwer Academic, Dordrecht, 1993.

[8] A. Korzeniowski and A. Oprisan, “Large deviations for additive functionals of Markov processes”, International Journal of Pure and Applied Mathematics, vol. 53, pp. 441–459, 2009.

[9] M.T. Lacey and W. Philipp, “A note on the almost sure central limit theorem”, Statistics & Probability Letters, vol. 9, pp. 201–205, 1990.

[10] M Lifshits, “Almost sure limit theorem for martingales”, Limit Theorems in Probability and Statistics II, J. Bolyai Mathematical Society, pp. 367–390, 2002.

[11] F. Maaouia, “Principes d’invariance par moyennisation logarithmique pour les processus de Markov”, The Annals of Probability, vol. 29, pp. 1859–1902, 2001.

[12] S.P. Meyn and R.L. Tweedie, Markov Chains and Stochastic Stability, Springer, New York, 1993.

[13] A. Oprisan and A Korzeniowski, “Large deviations via almost sure clt for functionals of Markov processes”, Stochastic Analysis and Applications, vol. 30, pp. 933–947, 2012.

[14] P. Schatte, “On the central limit theorem with almost sure convergence”, Prob. Math. Statist. , vol. 11, pp. 237–246, 1991.